MATH 1132Q Module IV Review

A review of Parametric & Polar Equations

Arc Length

Aka f'(y)

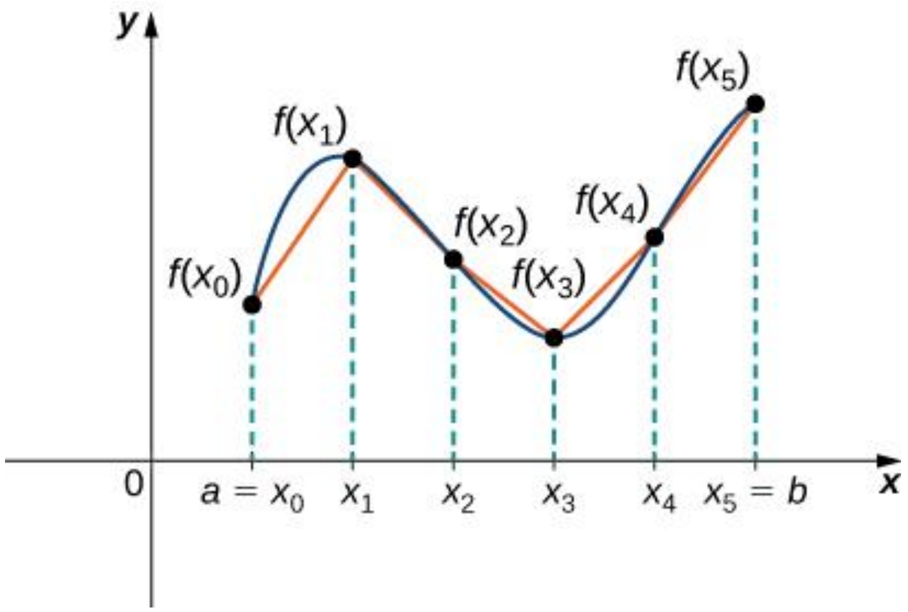
As a function of x:

$$S = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

As a function of y:

$$S = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

- What does this formula mean?
  - Arc length is essentially the **sum** of an infinite amount of **tiny line segments** along a curve
  - Can be expressed as a function of x or y, **depending on which results in a simpler integral**



Arc Length - Example

$$S = \int_a^b \sqrt{1 + (f'(x))^2}\, dx \qquad S = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

- Find arc length on the curve $y = x^{2/3}$ from x = 0 to x = 8.

You could try plugging this function into the f'(x) formula, but you'll end up with a no-so-fun integral to evaluate →

$$S = \int_0^8 \sqrt{1 + \frac{4}{9}x^{-2/3}}\, dx$$

So, try rewriting the function with respect to y instead and plug into the f'(y) formula →

$$y = x^{2/3} \longrightarrow y = (\sqrt[3]{x})^2 \longrightarrow x = y^{3/2}$$

⚠️ Remember to change bounds when switching WRT x to y ⚠️

When $x = 0$, $y = 0$

When $x = 8$, $y = 8^{2/3} = 4$

⟶ New bounds: y = 0, y = 4

Arc Length – Example continued

$$S = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Now that we are in terms of y, we can find f'(y) and plug into formula

$$x = y^{3/2} \longrightarrow \frac{dx}{dy} = \frac{3}{2} y^{1/2} \longrightarrow S = \int_0^4 \sqrt{1 + \left(\tfrac{3}{2} y^{1/2}\right)^2}\, dy$$

Simplifying:

$$S = \int_0^4 \sqrt{1 + \frac{9}{4} y}\, dy$$

This integral can be evaluated with u-sub, where:

$$u = 1 + \frac{9}{4} y$$

$$du = \frac{9}{4} dy$$

$$y = 0 \Rightarrow u = 1$$

$$y = 4 \Rightarrow u = 10$$

$$S = \frac{4}{9} \int_1^{10} \sqrt{u}\, du$$

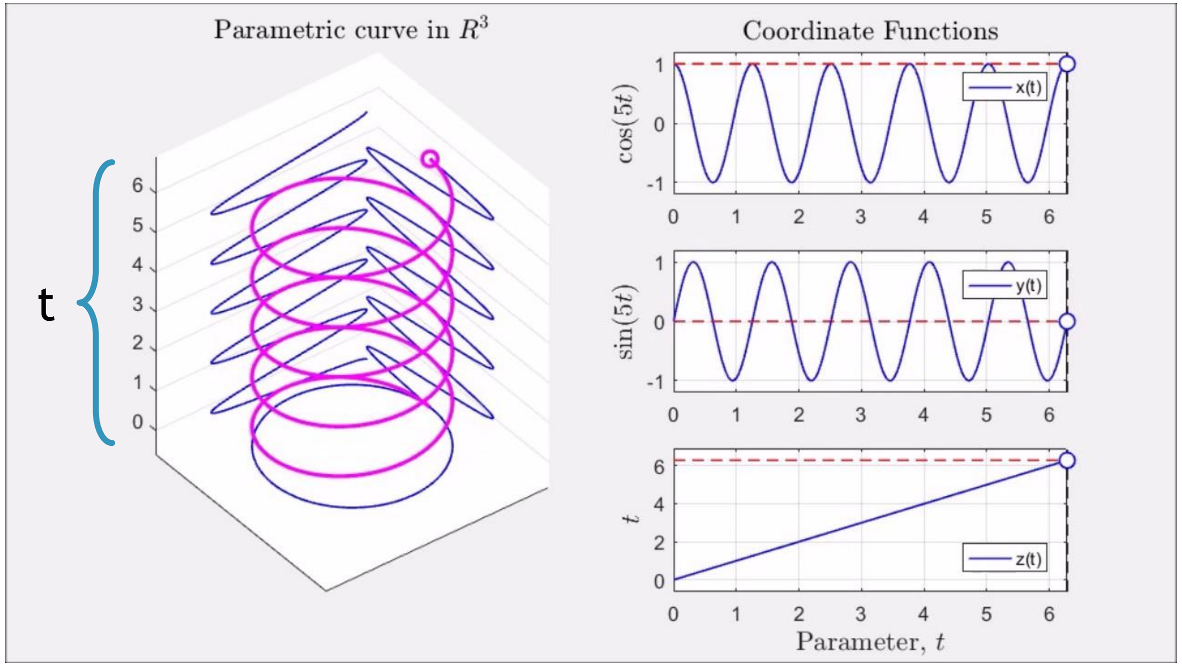
$$= \frac{4}{9} \cdot \frac{2}{3} \left[u^{3/2}\right]_1^{10}$$

$$\boxed{= \frac{8}{27} \left(10^{3/2} - 1\right)}$$

Parametric Equations

- What are they?
  - Equations that describe curves that can't be described by x & y only
  - Introduce a third variable, t (think t = time and x & y = position)

- How do they work?
  - For PE's, x & y are functions of a parameter, t
  - For each value of t, you get a corresponding point ((x(t), y(t))
  - An interval of t values (like t = [1, 3]) produces a parametric curve



Parametric Equations - Example

- Consider the parametric curve: $(x, y) = (t^2, 2t^3), \quad 0 \leq t \leq 1$

  Write **x in terms of y**

  (i.e. eliminate the parameter t)

$$\underbrace{t^2}_{= x(t)}, \quad \underbrace{2t^3}_{= y(t)}$$

"Eliminate t" means to solve for t using either x(t) or y(t) and substitute it into the other function.

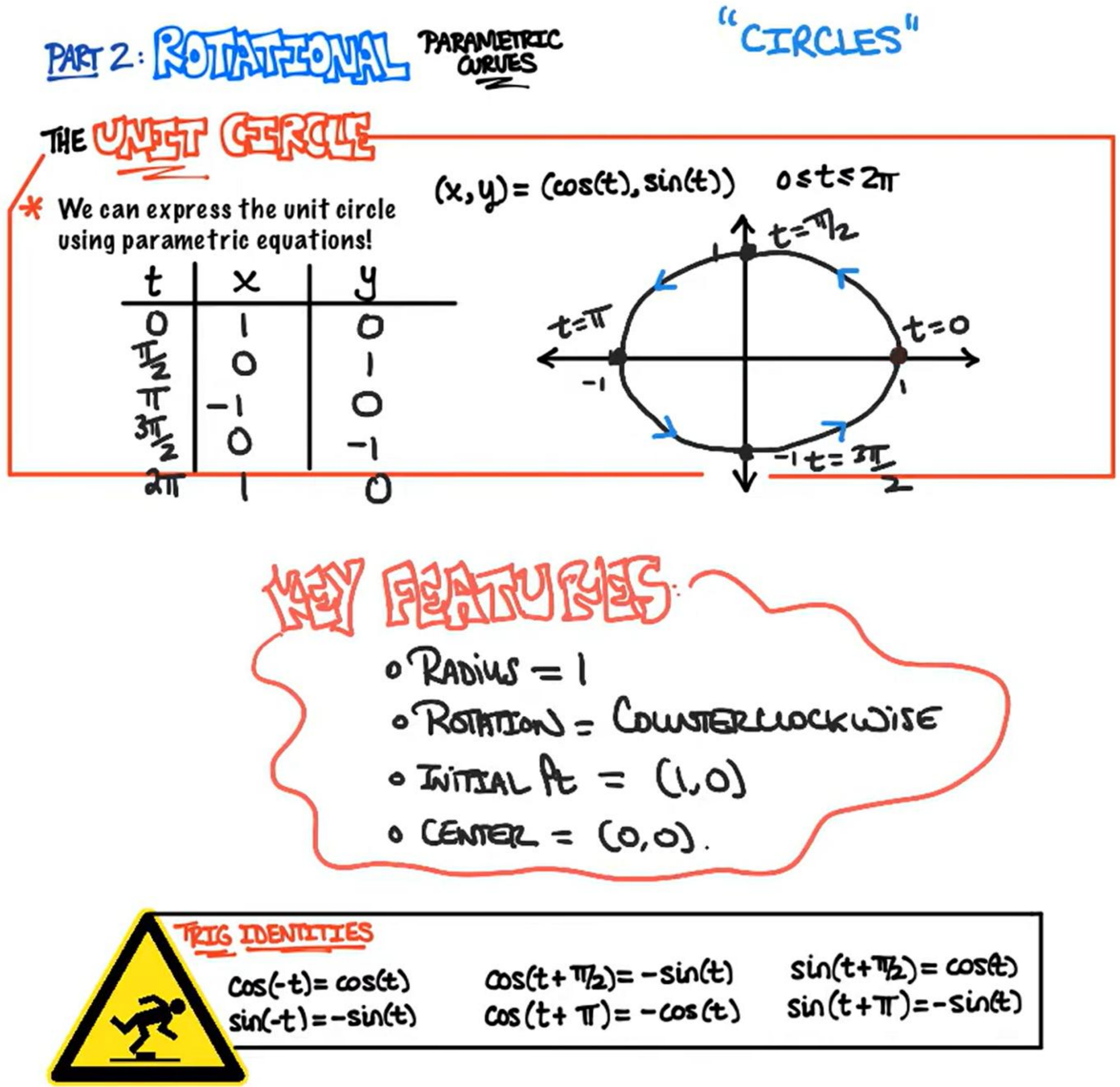
Let's start with y(t), solve for t, and plug result into x(t).

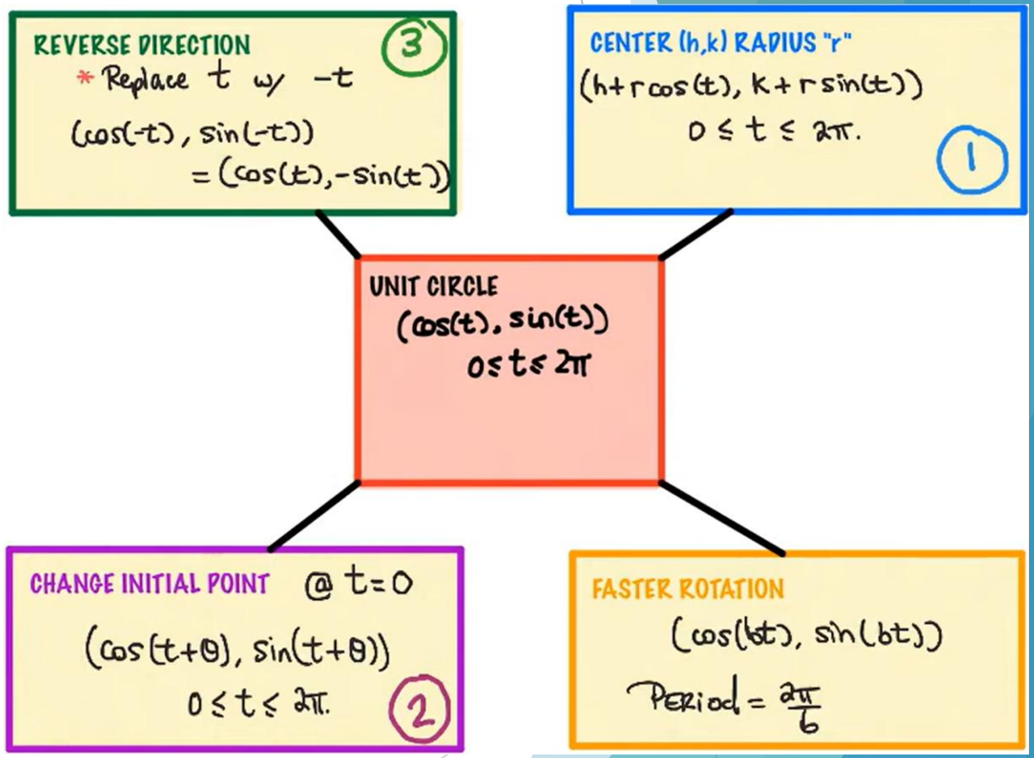
$$y = 2t^3 \longrightarrow \frac{y}{2} = t^3 \longrightarrow t = \left(\frac{y}{2}\right)^{1/3}$$

$$x = t^2 = \left(\left(\frac{y}{2}\right)^{1/3}\right)^2$$

$$\boxed{x = \left(\frac{y}{2}\right)^{2/3}}$$

Rotational Parametric Curves





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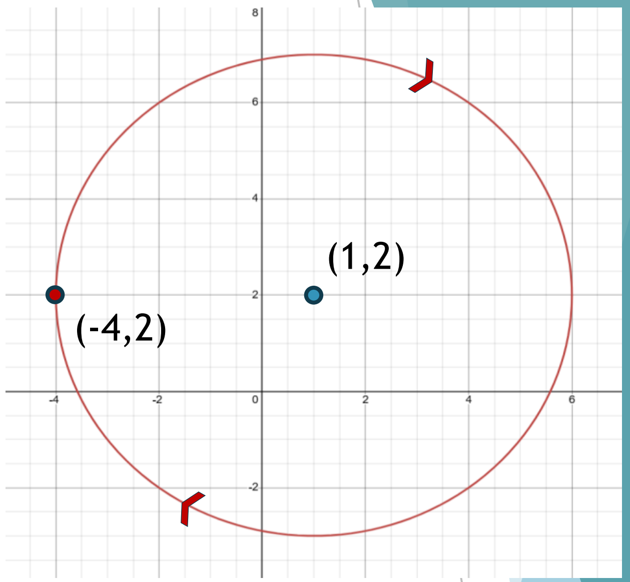
Rotational Parametric Curves – Ex.

Always helpful to draw a circle!



- Find the parameterization of a circle with a center (1,2) radius 5, oriented clockwise, with initial point (-4, 2)

For circles, always start with these & plug in the center and radius:

$$x = h + r\cos t \qquad y = k + r\sin t \quad \longrightarrow \quad x = 1 + 5\cos t \qquad y = 2 + 5\sin t$$

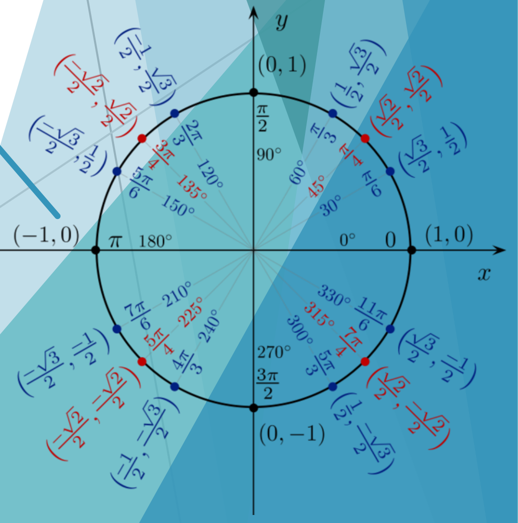
Next, we deal with the initial point. A circle centered at the origin has the **initial point (1,0)**, so in this case the default initial point is (6,2). We need to change that to (-4,2).

To do this, we **set x & y equal** to the coordinates of the **desired initial point (-4,2).**

$$1 + 5\cos t = -4 \Rightarrow \cos t = -1$$
$$2 + 5\sin t = 2 \Rightarrow \sin t = 0$$

What value for t do these cases both occur at?
From unit circle, **t = π**

$$t \to t + \pi$$



Now that we have our t value to shift our initial point, all we have left is our circle orientation. By default, the circle is **oriented CCW**. To make it go CW, we can **negate the sin(t) term**.

Sign flipped to make CW

$$\boxed{x(t) = 1 + 5\cos(t+\pi), \quad y(t) = 2 - 5\sin(t+\pi)}$$

Calculus with Parametric Curves

- What is the slope of a parametric curve?

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Horizontal tangent occurs when this = 0

Vertical tangent occurs when this = 0

Equation of a tangent line:

$$y - y_0 = m(x - x_0)$$

y(t)

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

x(t)

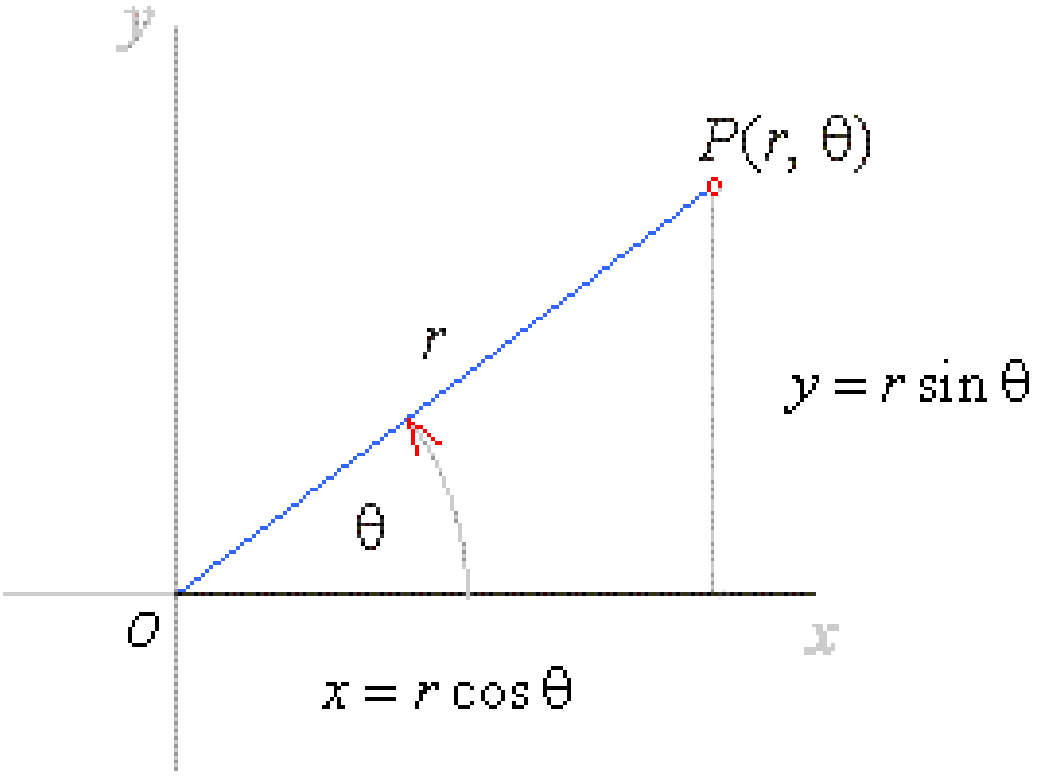
Problems you may encounter:

- Find equation of tangent line on parametric curve
- Find direction of increasing t values
- Find coordinates of points where parametric curve is horizontal/vertical

Treat these like basic Calc I problems—only this time there is a third variable "t"!

Polar Coordinates

- What's the point?
  - Polar coordinates are simply another way to express coordinates that makes dealing with **circular or rotational situations** much easier
  - Think: Cartesian coordinates (x, y) → Linear "language"
    Polar coordinates (r, θ) → Rotational "language"



Convert Cartesian to Polar

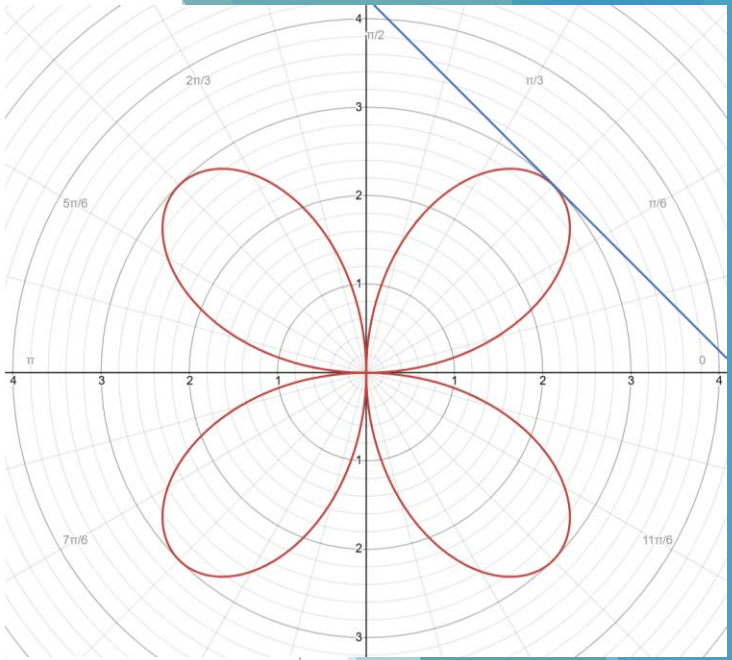
$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\frac{y}{x}$$

note: may need to add 180º

Polar Coordinates - Example

- Find the equation of the tangent line to the graph
  r = 3sin(2θ) when θ = π/4



Looking at the graph (and given the problem), we are in **polar coordinates**. Before we do anything, we must switch from **cartesian to polar** using:

$$x = r\cos\theta \qquad y = r\sin\theta \longrightarrow x = 3\sin(2\theta)\cos\theta \qquad y = 3\sin(2\theta)\sin\theta$$

Knowing the equation of a tangent line, we must find **y0, x0, and m.**

$$y - y_0 = m(x - x_0)$$

Let's find y0 and x0 at **θ = π/4:**

$$x0 = 3\sin\left(2\cdot\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) = 3\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right) = 3(1)\left(\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2}$$

$$y0 = 3\sin\left(2\cdot\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) = 3\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right) = 3(1)\left(\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2}$$

$$\boxed{y - \frac{3\sqrt{2}}{2} = -1\left(x - \frac{3\sqrt{2}}{2}\right)}$$

Finally, let's find the slope value at **θ = π/4** and plug it into the tangent line equation:

$$\mathbf{m} = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

Derivative of y with resp. to θ

Derivative of x with resp. to θ

Product rule:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

$$\frac{dy}{d\theta} = 6\cos(2\theta)\sin\theta + 3\sin(2\theta)\cos\theta$$

$$\frac{dx}{d\theta} = 6\cos(2\theta)\cos\theta - 3\sin(2\theta)\sin\theta$$

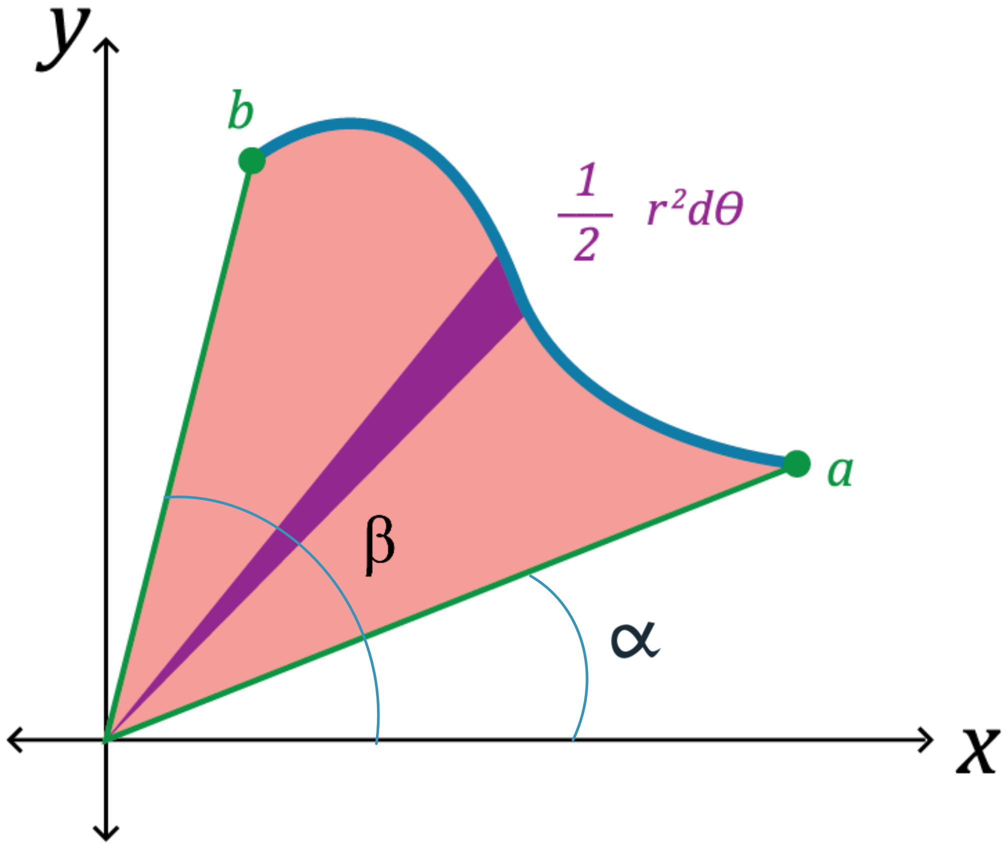
θ = π/4, so

$$= \cancelto{0}{6\cos\left(2\cdot\frac{\pi}{4}\right)}\cos\left(\frac{\pi}{4}\right) - 3\sin\left(2\cdot\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)$$

$$= \cancelto{0}{6\cos\left(2\cdot\frac{\pi}{4}\right)}\sin\left(\frac{\pi}{4}\right) + 3\sin\left(2\cdot\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)$$

$$\frac{dy/d\theta}{dx/d\theta} = \frac{3\cdot1\cdot\sqrt{2}/2 + 0}{-3\cdot1\cdot\sqrt{2}/2 + 0} = \boxed{-1}$$

Area in Polar Coordinates



$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left(f(\theta)\right)^2 d\theta$$

$$r^2 = x^2 + y^2$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

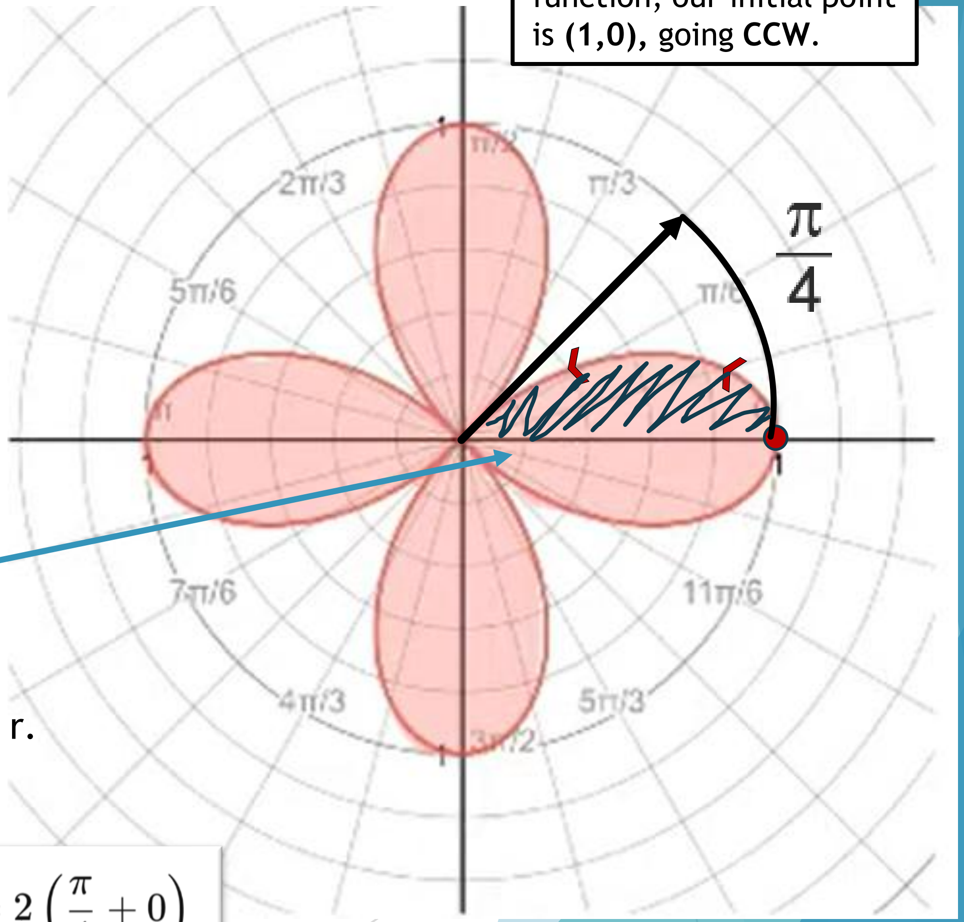
Area in Polar Coordinates – Example (Single Curve)

- Find the area of the polar region r = cos(2θ)

(= f(θ))

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta \quad \longrightarrow \quad A = \frac{1}{2}\int_{\alpha}^{\beta} \cos^2(2\theta)\, d\theta$$

*Remember that since **cos** composes the function, our initial point is **(1,0)**, going **CCW**.



*But what are the bounds? This one is tricky—we must recognize that the geometry of this curve provides a certain symmetry that will allow us to use a shortcut.

Here's the shortcut:
We use **0 to π/4** as our bounds and **multiply the entire integral by 8**, since there are 8 identical "**slices**" (shaded with scribbles) within the total area.

So now we get: $A = 8 \cdot \frac{1}{2}\int_{0}^{\frac{\pi}{4}} \cos^2(2\theta)\, d\theta$, which represents the entire area of r.

Using the identity $\cos^2(2\theta) = \frac{1+\cos(4\theta)}{2}$ to simplify, we get:

$$A = 2\int_{0}^{\frac{\pi}{4}} (1+\cos(4\theta))\, d\theta$$

Splitting the integral and evaluating we get:

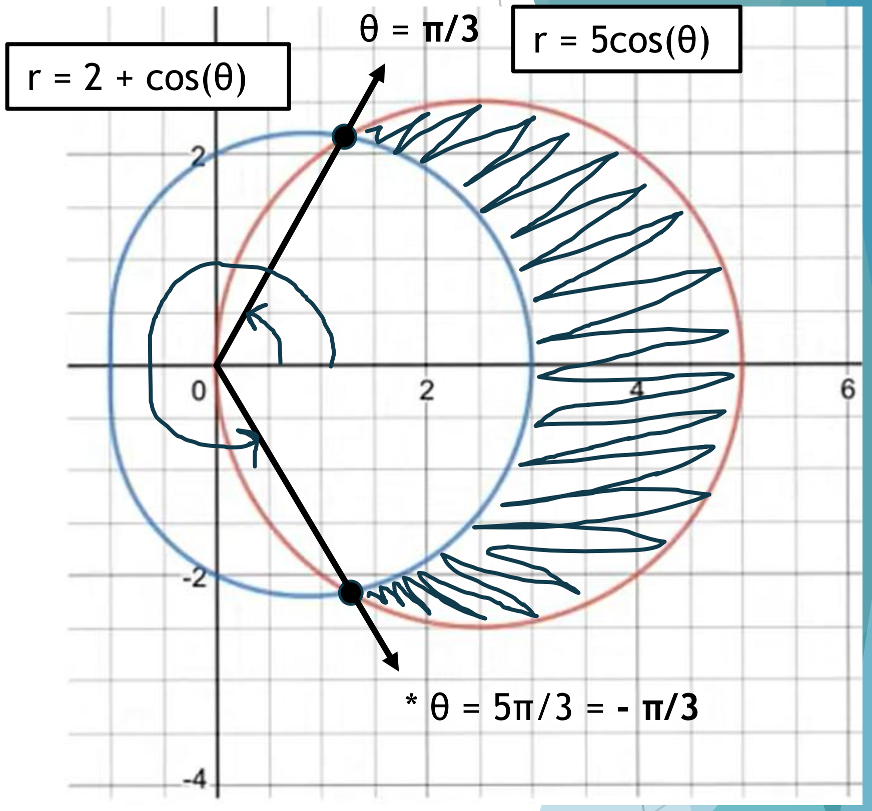
$$A = 2\left(\frac{\pi}{4} + 0\right)$$

$$A = \frac{\pi}{2}$$

Area in Polar Coordinates – Example (Intersecting Curves)

- **Set up** the integral used to find the area **inside** of the polar curve **r = 5cos(θ) & outside** of **r = 2 + cos(θ).**

First: Identify **which function is which** by plugging in a value for θ into each curve and comparing with the graph (ex. **θ = 1**)
Then: Identify **area of interest** (shaded with scribbles) and the **intersection points** on the graph



Next: Set the curves equal to one another and solve for θ.

$$5\cos\theta = 2 + \cos\theta$$

$$4\cos\theta = 2 \quad \Rightarrow \quad \cos\theta = \frac{1}{2}$$

* Where does cos = ½? At **θ = π/3, 5π/3**

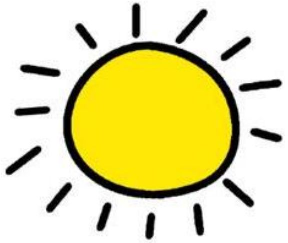
Setup:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$$

Area between curves = large curve – small curve
so,
Shaded Area = red curve - blue curve

$$\frac{1}{2}\int_{-\pi/3}^{\pi/3} (5\cos\theta)^2\, d\theta - \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2+\cos\theta)^2\, d\theta = \boxed{\frac{1}{2}\int_{-\pi/3}^{\pi/3} \left[(5\cos\theta)^2 - (2+\cos\theta)^2\right] d\theta}$$

** Use trig identity to evaluate



Good luck studying!