MATH 2410Q – Solving Basic Differential Equations

Hey hey y'all, it's Akeva! I'm going to be over some of the preliminary ways to solve differential equations. Previously, we only focused on solving equations indirectly with solution curves or taking given general solutions and specifying them to initial values. Now, we'll be going over the following topics:

- Separable Equations
  - Solution by integration
- Linear Equations
  - Standard form
  - Integrating Factor
- Exact Equations
  - Solving exact equations
- Solutions by Substitution
  - Bernoulli Equations

To set up this document, as my other guides, underlined words are sections, italicized words are subsections, red refers to definitions, and purple are for problems I do.

<u>Separable Equations</u>

Definitions

Before we dive into solving these equations, let's define what a separable equation is so we can recognize it more easily. Separable equations are first order ODEs which are solvable by rearranging terms to separate variables. Overall, their forms are the following:

$$\frac{dy}{dx} = f(x)g(y)$$

This can be rearranged such that:

$$\frac{1}{g(y)}dy = f(x)dx$$

This happens because we can almost treat the derivative are a separable numerator and denominator.

In this way, we can solve the differential equation to either get an implicit or explicit solution. An implicit solution is one that is not in the form $y = y(x)$, while an explicit solution follows that form. I think the best way to show these concepts are through problems so I will do a couple that will further practice concepts we went over such as IVPs and intervals of validity.

These examples will come from the Paul's Online Notes:

The first problem is "Solve the following differential equation and determine the interval of validity for the solution".

$$\frac{dy}{dx} = 6y^2x \qquad y(1) = \frac{1}{25}$$

Let's separate the variables, since we have confirmed that y and x are related through multiplication.

$$\frac{1}{y^2}dy = 6xdx$$

Now, we take the integral of both sides:

$$\int \frac{1}{y^2}dy = \int 6x\,dx$$

$$\int y^{-2}dy = \int 6x\,dx$$

$$-\frac{1}{y} + C_1 = 3x^2 + C_2$$

$$-\frac{1}{y} = 3x^2 + C$$

You can just essentially combine the two constants as they would just be a single constant, regardless of their values. And note that the form above is implicit because that is not in the right form. Now, let's determine C for the initial value given.

$$-\frac{1}{\frac{1}{25}} = 3(1)^2 + C$$

$$-25 = 3 + C$$

$$C = -28$$

Now, we can plug it in that equation and find the explicit form:

$$-\frac{1}{y} = 3x^2 - 28$$

$$\left(28 - 3x^2 = \frac{1}{y}\right)^{-1}$$

$$\frac{1}{28 - 3x^2} = y$$

We now have to determine the interval of validity. Since we have a denominator, let's focus there first. Let's set the denominator equal to zero:

$$28 - 3x^2 = 0$$

$$3x^2 = 28$$

$$x^2 = \frac{28}{3}$$

$$x = \pm\sqrt{\frac{28}{3}}$$

Therefore, the intervals we have consider are: $\left(-\infty, -\sqrt{\frac{28}{3}}\right)$, $\left(-\sqrt{\frac{28}{3}}, \sqrt{\frac{28}{3}}\right)$, and $\left(\sqrt{\frac{28}{3}}, \infty\right)$.

Based on the initial value given, we see the interval with 1 is , $\left(-\sqrt{\frac{28}{3}}, \sqrt{\frac{28}{3}}\right)$, making that the interval of validity.

Awesome! Let's try another problem to solidify the above technique. I'll choose something a little harder.

"Solve the following IVP."

$$\frac{dy}{dt} = e^{y-t}\sec(y)\,(1 + t^2) \qquad y(0) = 0$$

Notice, how there is no x, so we have to separate y and t. You will know by just looking at how the derivative is defined. Okay, now let's separate derivatives:

$$\frac{dy}{dt} = e^y e^{-t}\sec(y)\,(1 + t^2)$$

$$\frac{1}{e^y\sec(y)}dy = e^{-t}(1 + t^2)dt$$

Perfect! In this case, both sides look really ugly to solve but let's address them one at a time. Let's simplify the left-hand side (LHS) by using a reciprocal trig identity:

$$\sec(y) = \frac{1}{\cos(y)}$$

$$e^{-y}\cos(y)\,dy = e^{-t}(1 + t^2)dt$$

That got rid of the fraction, but we can't readily take the derivative with any basic techniques. Based on the fact that both sides are based on functions of y and t (that are not one power away from the other) for the LHS and RHS, respectively, we can use integration by parts. As a refresher, integration by parts takes the form:

$$\int u dv = uv - \int v du$$

Now, we apply this to our situation on the LHS:

$$u = e^{-y}, \quad dv = \cos(y), \quad du = -e^{-y}, \quad v = \sin(y)$$

$$e^{-y}\sin(y) - \int -\sin(y)\, e^{-y}\, dy$$

I chose dv to be the cosine term because, I know that the derivative of an exponential is just itself over and over. However, with the cosine, the derivative needs to be taken twice to get the cosine term back. This will be helpful when solving this kind of integration by parts. As a rule of thumb, if you see an exponential and a sine or cosine, set dv (both times, hint hint) to the trig term. Here, we will need to do another integration by parts (I know, our favorite), where parts are defined as:

$$u = \mathrm{e}^{-y}, \quad dv = -\sin(y), \quad du = -\mathrm{e}^{-y}, \quad v = \cos(y)$$

$$e^{-y}\sin(y) - \left[\cos(\mathrm{y})\,\mathrm{e}^{-\mathrm{y}} - \int -e^{-y}\cos(y)\, dy\right]$$

$$e^{-y}\sin(y) - \left[\cos(y)\, e^{-y} + \int e^{-y}\cos(y)\, dy\right]$$

Now, we see the last term is what we started with. If you remember, technically, the last equation is set equal to that term, so:

$$\int e^{-y}\cos(y)\, dy = e^{-y}\sin(y) - \left[\cos(y)\, e^{-y} + \int e^{-y}\cos(y)\, dy\right]$$

Now, put the repeating terms on one side:

$$2\int e^{-y}\cos(y)\, dy = e^{-y}\sin(y) - \cos(y)\, e^{-y}$$

$$\int e^{-y}\cos(y)\, dy = \frac{e^{-y}\sin(y) - \cos(y)\, e^{-y}}{2} + C_1$$

Okay! Now, we have the integral for the LHS. Let's do the RHS now. Try the process on your own first and then check with my work!

$$\int e^{-t}(1+t^2)dt$$

$$u=(1+t^2),\quad dv=e^{-t},\quad du=2t,\quad v=-e^{-t}$$

I chose u to be the polynomial term because ultimately, we will take the derivative of it for the inner integral, which reduces it power. Since we are constantly reducing the power of the polynomial, we will eventually get it down to a power of zero, making us not run into an infinite loop of integrals. Just use this as a rule of thumb.

$$-(t^2+1)e^{-t}-\int -e^{-t}(2t)\,dt$$

We need to do another integration by parts, just as we did previously:

$$u=2t,\quad dv=-e^{-t},\quad du=2,\quad v=\ e^{-t}$$

$$-(t^2+1)e^{-t}-\left(2te^{-t}-\int 2e^{-t}\,dt\right)$$

The integral inside is something we can solve with basic integration techniques, so let's do it!

$$-(t^2+1)e^{-t}-\left(2te^{-t}-(-2e^{-t})\right)$$

$$-(t^2+1)e^{-t}-(2te^{-t}+2e^{-t}+C_2)$$

$$-(t^2+1)e^{-t}-(2te^{-t}+2e^{-t}+C_2)$$

$$-t^2e^{-t}-e^{-t}-(2te^{-t}+2e^{-t}+C_2)$$

$$-t^2e^{-t}-e^{-t}-2te^{-t}-2e^{-t}-C_2$$

$$-t^2e^{-t}-2te^{-t}-3e^{-t}-C_2$$

Okay, let's put everything together:

$$\frac{e^{-y}\sin(y)-\cos{(y)}e^{-y}}{2}+C_1=-t^2e^{-t}-2te^{-t}-3e^{-t}-C_2$$

Since C_1 and C_2 are both constants, we can just combine them into an arbitrary constant, C. This makes the above equation now:

$$\frac{e^{-y}\sin(y)-e^{-y}\cos(y)}{2}=-t^2e^{-t}-2te^{-t}-3e^{-t}+C$$

Awesome! We took the derivatives of both sides, but now we need to solve the IVP and solve for C. Let's solve for it by plugging in 0 for both y and t (remember the initial condition given).

$$\frac{e^{-0}\sin(0)-e^{-0}\cos(0)}{2}=-0^2e^{-0}-2(0)e^{-0}-3e^{-0}+C$$

$$\frac{0-1}{2} = -3 + C$$

$$-\frac{1}{2} = -3 + C$$

$$-\frac{1}{2} + 3 = C$$

$$-\frac{1}{2} + \frac{6}{2} = C$$

$$\frac{5}{2} = C$$

Now, let's plug this into our original equation for the implicit solution!

$$\frac{e^{-y}\sin(y) - e^{-y}\cos(y)}{2} = -t^2e^{-t} - 2te^{-t} - 3e^{-t} + \frac{5}{2}$$

You can totally simplify if you want by grouping left terms on both sides. Depending on your professor, the above can be an acceptable solution.

For more practice with separable equations, use the following resources below:

- Organic Chemistry Tutor
- Professor Leonard (1 hr and 32 mins)
- Michel van Biezen (playlist)
- Brprp calc basics

Alrighty! We've done some of our first solving of differential equations. But there are many more ways to solve them. Let's move on to linear equations.

Linear Equations

In the basics of Differential Equations guide, I mentioned the concept of linear equations, but not how to solve them. To expand upon this more, I need to define some other concepts. One of them being standard form, which I indirectly introduced in the last guide. Standard form of a first-order linear differential equation separates the derivative term from $p(t)$ and $q(t)$, which are continuous equations and takes the below form:

$$\frac{dy}{dx} + p(t)y = g(t)$$

To solve these types of equations, we have to assume. The assumption is that there exists a function, $\mu(t)$, also known as an integrating factor, that will satisfy the following equation when multiplied to all terms in the equation:

$$\mu(t)p(t) = \mu'(t)$$

We don't need to get into the weeds of this. However, do know that $\mu(t)$ is defined as:

$$\mu(t) = e^{\int p(t)dt}$$

With this fact, the full general solution for first-order linear differential equations is:

$$y(t) = \frac{\int \mu(t)g(t)dt + c}{\mu(t)}$$

Where $c = \frac{c}{k}$ (where c and k are both constants from parts of the derivation I did not show).

Overall, to solve these types of equations, you use the following steps:

1. Write the equation in standard form.
2. Determine the integrating factor, using the equation above.
3. Multiply all terms in the differential equation by the integrating factor and verify that the LHS becomes the product rule and write it that way
4. Integrate both sides
5. Solving for the solution $y(t)$

An example of this can be found in the following differential equation can be found from Paul's Online Notes:

"Find the solution to the following differential equation.

$$\frac{dv}{dt} = 9.8 - 0.196v$$

First, we need to put the equation into standard form and find the integrating factor defined as:

$$\mu(t) = e^{\int p(t)dt}$$

The equation should now look like the following:

$$\frac{dv}{dt} + 0.196v = 9.8$$

Now that we separated out our $p(t)$, we can plug and chug:

$$\mu(t) = e^{\int 0.196dt}$$

$$\int 0.196dt = 0.196t$$

$$\mu(t) = e^{0.196t}$$

Perfect, now we need to multiply this to all terms in the equation and integrate both sides as normal:

$$e^{0.196t}\frac{dv}{dt} + 0.196v(e^{0.196t}) = 9.8(e^{0.196t})$$

(A quick verification that the LHS is the product rule $(\mu(t)y(t))'$):

$$\big(\mu(t)y(t)\big)' = (e^{0.196t} * v)$$

$$= 0.196e^{0.196t}v + \frac{dv}{dt}e^{0.196t}$$

Now rewriting the product rule on the LHS and then integrating:

$$\int (e^{0.196t}v)'dt = \int 9.8(e^{0.196t})dt$$

$$e^{0.196t}v + k = \frac{9.8}{0.196}e^{0.196t} + c$$

$$e^{0.196t}v = \frac{9.8}{0.196}e^{0.196t} + c - k$$

$$e^{0.196t}v = \frac{9.8}{0.196}e^{0.196t} + c$$

$$\frac{e^{0.196t}v = \frac{9.8}{0.196}e^{0.196t} + c}{e^{0.196t}}$$

$$v(t) = \frac{9.8}{0.196} + ce^{-0.196t}$$

We weren't given initial conditions so we cannot solve for c in this case.

Alright, let's try this with an initial value problem. Let's say we were given the same problem but given an initial value of $v(0) = 48$. Now, we can take our result from before and solve for c:

$$48 = \frac{9.8}{0.196} + ce^{-0.196(0)}$$

$$48 = \frac{9.8}{0.196} + c$$

$$48 = 50 + c$$

$$48 - 50 = c$$

$$c = -2$$

This makes the actual solution for the given IVP:

$$v(t) = \frac{9.8}{0.196} - 2e^{-0.196t}$$

For more practice with solving linear first-order equations, refer to the following resources:

- Organic Chemistry Tutor
- Math with Professor V
- blackpenredpen
- Professor Leonard
- Paul's Online Notes

Exact Equations

Another type of differential equation that you will or should have been introduced to is an exact equation. To be specific, the exact equations take the following form:

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

Yes, there must be only those two terms and the ability to have zero on one side. Furthermore, there must be addition between the M and N terms, for this to work. The implicit solution to an exact differential equation is:

$$\Psi(x, y) = c$$

In order for these to be an actual solution, both M and N need to be continuous and have their first order derivatives also be continuous. Specifically, for there to be a solution the following must be true:

$$M_y = N_x$$

From there, to find $\Psi(x, y)$, you can use either of the following equations (it's really on what's easiest to integrate):

$$\Psi = \int M\,dx \quad OR \quad \Psi = \int N\,dy$$

This may seem like a lot, but the general workflow to solving these problems is:

1. Identifying M and N and checking that the differential equation is exact using:
$$M_y = N_x$$
2. Finding Ψ using the equations and taking care to have a "constant" of integration (will be a function aka h(y) or g(x)):
$$\Psi = \int M\,dx \quad OR \quad \Psi = \int N\,dy$$
3. Determine h(y) or g(x) using the fact that differentiating Ψ with respect to y must equal N and differentiating Ψ with respect to x must equal M
4. Fully determining Ψ and any initial condition
5. If asked, determine the interval of validity

Let's try out a problem so I can go through the steps and my thought process. As usual, this problem is taken from Paul's Online Notes:

"Find the solution for the following IVP.

$$2xy^2 + 4 = 2(3 - x^2y)y' \qquad y(-1) = 8$$

At first look, it may seem as though this isn't an exact differential equation due to the $+4$ term. However, think of it as a part of the M term, which is not associated with the derivative of the function. For clarity, we can rewrite the above equation in this way, by putting parenthesis around the M term and moving N over to the left with simplification:

$$(2xy^2 + 4) + (2x^2y - 6)y' = 0$$

Great, now before we even start to do calculations to solve the IVP, we need to double check that this is an exact equation. Let's do that:

$$M_y = \frac{d}{dy}(2xy^2 + 4) = 4xy$$

$$N_x = \frac{d}{dx}(2x^2y - 6) = 4xy$$

Since, they are equal, this is indeed an exact equation. Now let's set up finding Ψ. Since neither is bad, let's integrate M with respect to x:

$$\Psi(\mathrm{x,y}) = \int 2xy^2 + 4\,dx = x^2y^2 + 4x + g(y)$$

To find this $g(x)$, we need to differentiate Ψ with respect to y to set it equal to N:

$$\Psi_{\mathrm{y}} = 2x^2y + g'(y) = 2x^2y - 6$$

We can see that $g'(y) = -6$, making $g(y) = -6y$.

So,

$$\Psi(\mathrm{x,y}) = x^2y^2 - 6y + 4x + c$$

Now, we can solve the initial value problem by plugging it in to get:

$$(-1)^2(8)^2 - 6(8) + 4(-1)$$

$$= 64 - 48 - 4 = 12$$

This makes the solution:

$$x^2y^2 - 6y + 4x + 12 = 0$$

I hope that wasn't too bad and my logic made sense. If not, I'll link some resources and videos that can explain the concepts differently and give you more practice problems:

- Patrick J
- blackpenredpen
- Math with Professor V (~1 hr and 12 minutes)
- Professor Leonard
- LibreTexts Mathematics

Solutions by Substitution

At times, you cannot readily use the techniques I have mentioned thus far in this document. In these cases, some extra manipulation needs to be done to bring it down to a solvable form. I'm sure by looking at the title of this section, you can guess that it's a similar concept to u-substitution when taking the integral of a function. This allows for us to convert complex, non-separable equations into simpler, and more importantly, solvable forms. There are different types so we'll being going through them and some problems that involve them.

Homogenous Substitution

For this substitution, the differential equation needs to be in the following form:

$$y' = F\left(\frac{y}{x}\right)$$

As you can probably guess from the name, first-order differential equation that can be written in the above form are called homogenous equations. We can also determine if an equation is homogenous, we use the fact that a function $f(x, y)$ is homogenous of degree n if

$$f(tx, ty) = t^n f(x, y)$$

Furthermore, for a differential equation of the following form:

$$M(x, y)dx + N(x, y)dy = 0$$

It is homogenous if $M(x, y)$ and $N(x, y)$ are homogenous functions of the same degree.

Once we know the equation is homogenous and in the $y' = F\left(\frac{y}{x}\right)$ form, we use the substitution:

$$v(x) = \frac{y}{x}$$

Which can then be rewritten as:

$$y = xv$$

With this, we can take the derivative to find y' and use the fact that both y and v are functions of x, to get

$$y' = v + xv'$$

Now, we substitute these into the homogenous equation:

$$y' = F\left(\frac{y}{x}\right)$$

$$v + xv' = F(v)$$

$$xv' = F(v) - v$$

Hopefully this is starting to look familiar. Hint hint, it's a separable equation.

$$x\frac{dv}{dx} = F(v) - v$$

$$\frac{dv}{F(v) - v} = \frac{dx}{x}$$

This may seem like a lot but once you get the hang of it, it's simple. Let's start off with identifying if an equation is homogenous. I'll be drawing from LibreTexts Mathematics.

"Determine if $f(x, y) = x^5 + x^2y^3$ is a homogenous function and, if so, of what degree."

Remember that if the equation is homogenous, it follows the following form:

$$f(tx, ty) = t^n f(x, y)$$

So, let's start with plugging it t.

$$(tx)^5 + (tx)^2(ty)^3$$

$$t^5x^5 + (t^2x^2)(t^3y^3)$$

$$t^5x^5 + t^5(x^2y^3)$$

$$t^5(x^5 + x^2y^3)$$

Therefore, it is a homogenous equation of degree 5 because t^5 is multiplied to the original function.

Now, let's try solving a differential equation with a homogenous substitution. This problem is from Paul's Online Notes:

"Solve the following IVP:

$$xyy' + 4x^2 + y^2 = 0 \quad y(2) = -7, \;\; x > 0$$

Looking at this equation, we don't see a dx or dy, so we have to try to convert the equation to the following form:

$$y' = F\left(\frac{y}{x}\right)$$

So, let's divide the equation by x^2 since that can give us $\frac{y}{x}$ with derivative term and even the last term in the equation:

$$\frac{xyy'}{x^2} + \frac{4x^2}{x^2} + \frac{y^2}{x^2} = 0$$

$$\frac{yy'}{x} + 4 + \left(\frac{y}{x}\right)^2 = 0$$

$$\frac{yy'}{x} = -4 - \left(\frac{y}{x}\right)^2$$

Now, let's use the conclusions from above to do the substitution:

$$v = \frac{y}{x}, \qquad y = xv, \qquad y' = v + xv'$$

$$v(v + xv') = -4 - v^2$$

$$v^2 + xvv' = -4 - v^2$$

$$xv\frac{dv}{dx} = -4 - 2v^2$$

$$x\frac{dv}{dx} = \frac{-4 - 2v^2}{v}$$

$$\frac{v}{-4-2v^2}dv = \frac{dx}{x}$$

$$\int \frac{v}{4+2v^2}dv = \int -\frac{dx}{x}$$

The left side is going to need u-substitution since the numerator can be a derivative of the bottom v term:

$$u = 4+2v^2, \quad \frac{du}{dv} = 4v, \quad dv = \frac{du}{4v}$$

$$\int \frac{v}{u}\left(\frac{du}{4v}\right) = \int \frac{du}{4u} = \frac{1}{4}\int \frac{du}{u} = \frac{1}{4}\ln u + C_1$$

Now let's do the RHS:

$$\int \frac{dx}{x} = \ln x + C_2$$

So, we now have:

$$\frac{1}{4}\ln(4+2v^2) + C_1 = -\ln x + C_2$$

$$\frac{1}{4}\ln(4+2v^2) = -\ln x + C$$

$$\ln(4+2v^2)^{\frac{1}{4}} = -\ln x + C$$

$$\ln\left((4+2v^2)^{\frac{1}{4}}\right) = -\ln x + C$$

$$e^{\ln\left((4+2v^2)^{\frac{1}{4}}\right)} = e^{\ln(x^{-1})+C}$$

$$(4+2v^2)^{\frac{1}{4}} = \frac{1}{x}e^C$$

$$(4+2v^2)^{\frac{1}{4}} = \frac{C}{x}$$

$$4+2v^2 = \frac{C^4}{x^4} = \frac{C}{x^4}$$

$$2v^2 = \frac{C}{x^4} - 4$$

$$v^2 = \frac{1}{2}\left(\frac{C}{x^4} - 4\right)$$

$$\frac{y^2}{x^2} = \frac{1}{2}\left(\frac{C}{x^4} - 4\right)$$

$$y^2 = \frac{x^2}{2}\left(\frac{C}{x^4} - 4\right)$$

$$y^2 = \frac{x^2}{2}\frac{(C - 4x^4)}{x^4}$$

$$y^2 = \frac{C - 4x^4}{2x^2}$$

Alright, now let's try to solve the IVP and leave the implicit form!

$$(7)^2 = \frac{C - 4(2)^4}{2(2)^2}$$

$$49 = \frac{C - 4(16)}{8}$$

$$392 = C - 4(16)$$

$$392 = C - 64$$

$$C = 456$$

So, the final solution is:

$$y^2 = \frac{456 - 4x^4}{2x^2}$$

As a note, if the equation comes in the form of $M(x,y)dx + N(x,y)dy = 0$, you need to verify that it is a homogenous equation by seeing if the M and N, separately are homogenous and then if they are of the same degree. After the confirmation, you can use the fundamental equations:

$$v = \frac{y}{x}, \qquad y = xv, \qquad y' = v + xv'$$

To make the equation separable.

For more practice with homogenous substitution, use the following resources (other than the ones I drew some problems from):

- Professor Leonard (~ 1 hr and 6 mins)
- blackpenredpen
- Patrick J

Bernoulli Equations

Another form of equations that are not readily solvable are differential equations that take the following form:

$$y' + p(x)y = q(x)y^n$$

Where $p(x)$ and $q(x)$ are continuous on the interval we are focused on and n is a real number. As you can infer from this section, the name of these differential equations of the above form is Bernoulli equations. A little note, though, if you see that n is 0 or 1, you know solve them as a linear equation and separable equation, respectively.

To solve for n greater than 1, we need to divide the differential equation by y^n:

$$y^{-n}y' + y^{-n}p(x)y = q(x)$$

$$y^{-n}y' + p(x)y^{1-n} = q(x)$$

As we did with homogenous substitution, we use a substitution ($v = y^{1-n}$) and differentiate this and substitute these into the equation. From there we can solve for v as a linear differential equation and then solve for y after substituting it back in. Okay, it's focus now on deriving the derivative of v first, before doing a problem:

$$v = y^{1-n}$$

$$v' = (1-n)y^{-n}y'$$

$$y' = \frac{y^n v'}{(1-n)}$$

We can plug this into the equation above to get our linear differential equation:

$$\frac{y^{-n}(y^n v')}{1-n} + p(x)y^{1-n} = q(x)$$

$$\frac{v'}{1-n} + p(x)v = q(x)$$

Perfect, let's try to do a problem with these facts in mind (from Paul's Online Notes, of course):

"Solve the following the IVP:

$$y' + \frac{4}{x}y = x^3y^2, \quad y(2) = -1, \quad x > 0$$

Remember what our first step was with these equations? It was dividing the equation by y^n. So, let's do that but with y^2.

$$\frac{y'}{y^2} + \frac{4}{x}\frac{y}{y^2} = x^3$$

$$\frac{y'}{y^2} + \frac{4}{xy} = x^3$$

Now, let's use our helpful equations:

$$v = y^{1-n}, \quad v' = (1-n)y^{-n}y'$$

And substitute v in to replace y:

$$v' = \frac{(1-2)y'}{y^2} = -\frac{y'}{y^2}$$

$$v = y^{1-2} = y^{-1} = \frac{1}{y}$$

So,

$$-v' + \frac{4}{x}v = x^3$$

$$v' - \frac{4}{x}v = -x^3$$

Now that we have our linear equation, we need to find our integrating factor, which is defined as:

$$\mu(x) = e^{\int p(x)dx}$$

For the equation:

$$\frac{dv}{dx} + p(x)v = g(x)$$

In this means that $p(v) = -\frac{4}{x}$ and

$$\mu(v) = e^{\int -\frac{4}{x}dx} = e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4} = x^{-4}$$

Now, we need to multiply this integrating factor to all the terms in the equation, getting:

$$x^{-4}\left(v' - \frac{4}{x}v = -x^3\right) = x^{-4}\frac{dv}{dx} - \frac{4}{x^{-5}}v = -\frac{1}{x}$$

The LHS is the product of $(x^{-4}v)'$ so, we can rewrite it and integrate both sides as we did before:

$$\int (x^{-4}v)'dx = \int -\frac{1}{x}dx$$

$$x^{-4}v + C_1 = -\ln x + C_2$$

$$x^{-4}v = -\ln x + C$$

$$v = -x^4 \ln x + Cx^4$$

Now, let's plug in y:

$$y^{-1} = -x^4 \ln x + Cx^4$$

And let's solve the IVP before writing the explicit solution:

$$\frac{1}{-1} = -(2)^4 \ln 2 + C(2)^4$$

$$-1 = -16 \ln 2 + 16C$$

$$16 \ln 2 - 1 = 16C$$

$$\ln 2 - \frac{1}{16} = C$$

So, now we can plug in C and find the explicit solution:

$$y^{-1} = -x^4 \ln x + \left(\ln 2 - \frac{1}{16}\right)x^4$$

$$y = \frac{1}{x^4\left(-\ln x + \ln 2 - \frac{1}{16}\right)} = \frac{1}{x^4\left(\ln\frac{2}{x} - \frac{1}{16}\right)}$$

Perfect, we are now done!

Linear Function Substitution

To my knowledge, this class of substitutions doesn't have a formal name as the others. I'll just call it linear function substitution for the sake of this document. However, there is still a clear form to it:

$$\frac{dy}{dx} = G(ax + by + c)$$

In these cases, the substitution that we should use is:

$$u = ax + by + c$$

Taking the derivative, we get:

$$u' = a + b\frac{dy}{dx}$$

Let's apply this to an example. This also comes from Paul's Online Notes:

"Solve the following IVP

$$y' - (4x - y + 1)^2 = 0, \quad y(0) = 2$$

Alright, this equation isn't in the exact form we need it to be in, so let's move somethings around:

$$y' = (4x - y + 1)^2$$

In this case, $u = 4x - y + 1$. Taking the derivative, we get

$$u' = 4 - y'$$

$$y' = 4 - u'$$

The derivative of y came from solving for it using the derivative of u equation. Now, we can substitute this into our equation:

$$4 - u' = u^2$$

$$4 - \frac{du}{dx} = u^2$$

This turns into a separable equation:

$$\frac{du}{dx} = 4 - u^2$$

$$\frac{du}{4 - u^2} = dx$$

We now take the integral of both sides:

$$\int \frac{du}{4 - u^2} = \int dx$$

$$\int \frac{du}{(2 - u)(2 + u)} = \int dx$$

For the LHS, we need partial derivatives (yippie…). If you need a review, use this resource. Alright, let's do it:

$$\frac{1}{(2 - u)(2 + u)} = \frac{A}{2 - u} + \frac{B}{2 + u}$$

$$1 = A(2+u) + B(2-u)$$

Solving for A, I will do $u = 2$.

$$1 = A(2+2)$$

$$1 = 4A$$

$$A = \frac{1}{4}$$

Now solving for B, $u = -2$.

$$1 = B(2 - -2)$$

$$1 = 4B$$

$$B = \frac{1}{4}$$

Now, we plug the partial decomposition to the LHS:

$$\int \frac{\frac{1}{4}}{2-u} + \frac{\frac{1}{4}}{2+u} du = \int dx$$

$$\frac{1}{4}(-\ln(2-u) + \ln(2+u)) = x + C$$

$$\frac{1}{4}\ln\left(\frac{2+u}{2-u}\right) = x + C$$

$$\ln\left(\frac{2+u}{2-u}\right) = 4x + C$$

$$e^{\ln\left(\frac{2+u}{2-u}\right)} = e^{4x+C}$$

$$\frac{2+u}{2-u} = Ce^{4x}$$

Let's plug in y here and solve for C for the IVP:

$$\frac{2+(4x-y+1)}{2-(4x-y+1)} = Ce^{4x}$$

$$\frac{2+4x-y+1}{2-4x+y-1} = Ce^{4x}$$

$$\frac{2+4x-y+1}{2-4x+y-1)} = Ce^{4x}$$

$$\frac{2+4(0)-(2)+1}{2-4(0)+2-1} = Ce^{4*0}$$

$$\frac{2-2+1}{2+2-1} = C$$

$$\frac{1}{3} = C$$

Now, we can plug that in and solve for y:

$$\frac{2+4x-y+1}{2-4x+y-1} = \frac{1}{3}e^{4x}$$

$$\frac{4x-y+3}{1-4x+y} = \frac{1}{3}e^{4x}$$

$$4x-y+3 = \frac{1}{3}e^{4x}(1-4x+y)$$

$$4x-y+3 = \frac{1}{3}e^{4x} - \frac{4x}{3}e^{4x} + \frac{y}{3}e^{4x}$$

$$3\left(4x-y+3 = \frac{1}{3}e^{4x} - \frac{4x}{3}e^{4x} + \frac{y}{3}e^{4x}\right)$$

$$12x-3y+9 = e^{4x} - 4xe^{4x} + ye^{4x}$$

$$12x+9-e^{4x}+4xe^{4x} = ye^{4x}+3y$$

$$12x+9-e^{4x}+4xe^{4x} = y(e^{4x}+3)$$

$$y = \frac{12x+9-e^{4x}+4xe^{4x}}{e^{4x}+3} = \frac{4x(3+e^{4x})+9-e^{4x}}{e^{4x}+3} = 4x + \frac{9-e^{4x}}{e^{4x}+3}$$

Okay, we are finally done! Hopefully seeing me do out these problems helped your understanding of the topic more!

If you need more practice outside of the places, I drew problems from, for substitution in general, use the following resources:

- Mike, the Mathematician (Linear substitution)
- Dr. Luke's Lectures (all types)
- blackpenredpen (all types – playlist)
- Professor Dave Explains (Homogenous and Bernoulli)

Alrighty, that is all I have regarding solving first-order differential equations with some introductory techniques. I hope this was helpful!