Laplace Overview/Review for MATH 2410

Hello everyone, this is Akeva! As you can probably tell by the document, I will be going over Laplace transforms, Inverse Laplace Transforms & Derivatives, and Operational Properties or DiffyQ. I will discuss key definitions and go over some resources that I think will help and how I approach these problems. I am probably the odd one out in saying this, but I loved DiffyQ, so I'll try my best to make this as simple for y'all as possible! :)

If you haven't seen my other guides, I typically use red for definitions that are fundamental to the topic at hand, and underlined words are headings for each topic.

Laplace Transform - Definition

Let's immediately jump into things and talk about Laplace Transforms. You might think its tedious and annoying, however, I would highly suggest that if you are in Engineering and have to take Electrical Circuits, please please please remember Laplace Transforms because it will make your lives so much easier when dealing with capacitors and inductors. I digress. Let's start with the question: what are Laplace Transforms?

Laplace transform: it's a transformation that are useful for solving linear ODE and defined as

$$\mathcal{L}[f(t)](s) = \int_0^\infty f(t)e^{-st}\,dt$$

where $f(t)$ is a function defined $t \geq 0$.

The equation is helpful if you do not have access to a cheat sheet on your exam to derive the necessary Laplace transforms of things. I would highly suggest memorizing the basic Laplace transforms if that is case. They can be found below (this one I found online):

The Laplace transforms of some common functions.

Function, $f(t)$	Laplace transform, $F(s)$	Function, $f(t)$	Laplace transform, $F(s)$
1	$\frac{1}{s}$	$e^{-at}\cos bt$	$\frac{s+a}{(s+a)^2+b^2}$
t	$\frac{1}{s^2}$	$\sinh bt$	$\frac{b}{s^2-b^2}$
t^2	$\frac{2}{s^3}$	$\cosh bt$	$\frac{s}{s^2-b^2}$
t^n	$\frac{n!}{s^{n+1}}$	$e^{-at}\sinh bt$	$\frac{b}{(s+a)^2-b^2}$
e^{at}	$\frac{1}{s-a}$	$e^{-at}\cosh bt$	$\frac{s+a}{(s+a)^2-b^2}$
e^{-at}	$\frac{1}{s+a}$	$t\sin bt$	$\frac{2bs}{(s^2+b^2)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	$t\cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
$\sin bt$	$\frac{b}{s^2+b^2}$	$u(t)$ unit step	$\frac{1}{s}$
$\cos bt$	$\frac{s}{s^2+b^2}$	$u(t-d)$	$\frac{e^{-sd}}{s}$
$e^{-at}\sin bt$	$\frac{b}{(s+a)^2+b^2}$	$\delta(t)$	1
		$\delta(t-d)$	e^{-sd}

Laplace Transforms – Properties

To lead each property, I'll ask some questions and answer them using the property.

1. How do we know when a Laplace transform exists?

Well, according to the existence theorem of Laplace transforms, if a function $f(t)$ is a piecewise, continuous (aka, a function that is consistent of different functions but at the point of discontinuity, one of functions is defined there), on every finite interval where $t \geq 0$ and satisfies $|f(t)| \leq Me^{at}$, then

$$\mathcal{L}[f(t)](s) = \int_0^\infty f(t)e^{-st}\,dt$$

for all $s > a$.

2. Are there multiple Laplace transforms for each function?

No, which is a characteristic of Laplace Transforms; they are unique (meaning for a particular function, there is only one Laplace transform aka one-to-one). This is due to the integral in the definition being bounded.

3. Are there simplifications we can make while making Laplace transforms?

Yes, there are many properties that can help simplify the process of finding a Laplace transform. They include:

- Linearity such that

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$$

- Simplifications of convolutions

$$\mathcal{L}[f(t) * g(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)]$$

    - For people like me who were very confused by convolutions, I'll give a basic definition of them: think of them as the amount of overlap between the two functions as the second function is shifted on the first

4. How can we make algebraic equations out of differential equations?

- Taking the Laplace transform of derivatives (typically the highest you go is the second derivative) takes the form of the following:

$$\mathcal{L}\left[f^{(n)}(t)\right](s) = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0)$$

Let's go over a practice problem with Laplace transforms and walk you through my thought process. Try it on your own before you follow with me.

Let's say I was given $y'' - y' - 2y = 0$. What's the Laplace transform?

I would immediately define each variable on the left-hand side, where my derivatives are. Based on the equation given under Question 4 above, I know the following:

$$Taking\ the\ Laplace\ transform\ of\ the\ left\ side: \mathcal{L}[y''] - \mathcal{L}[y'] - \mathcal{L}[2y]$$

$$\mathcal{L}[y''] = s^2\mathcal{L}[y] - sf(0) - f'(0)$$

$$\mathcal{L}[y'] = s\mathcal{L}[y] - sf(0)$$

$$\mathcal{L}[2y] = 2\mathcal{L}[y]$$

Putting everything together, we get:

$$s^2\mathcal{L}[y] - sf(0) - f'(0) - [\ s\mathcal{L}[y] - sf(0)] - [2\mathcal{L}[y]]$$

$$s^2\mathcal{L}[y] - sf(0) - f'(0) - s\mathcal{L}[y] + f(0) - 2\mathcal{L}[y]$$

$$\underline{\mathcal{L}[y](s^2 - s - 2) - sf(0) - f'(0) + f(0)}$$

Note: You will often get cancellations or major simplifications. If not, check your math.

For the right side, the Laplace transform of zero is 0. Putting both sides together, we get

$$\mathcal{L}[y](s^2 - s - 2) - sf(0) - f'(0) + f(0) = 0$$

$$\mathcal{L}[y](s^2 - s - 2) = sf(0) + f'(0) - f(0)$$

$$\mathcal{L}[y] = \frac{sf(0) + f'(0) - f(0)}{s^2 - s - 2}$$

If you need more practice with Laplace transforms, try the following resources:

- Practice Problems with Answers
- Steven Bruton
- SkanCity Academy

Inverse Laplace Transform - Definition

Awesome! We are one step closer to being able to use Laplace transforms to solve differential equations. Now we just need to do figure out go the other way or the inverse Laplace transform, when we get the point above, to get y.

So, what's an inverse Laplace transform? It's defined as the following:

$$f(t) = \mathcal{L}^{-1}[F(s)]$$

where $F(s)$ is the Laplace transform.

Honestly, the longest parts of these problems come here and are often a source of where mistakes can occur. I am going to go through the properties and continue walking through the above problem and give tips as I go.

Inverse Laplace Transform – Properties

- Linearity

Let's say we are given two Laplace transforms, $F(s)$ and $G(s)$, then

$$\mathcal{L}^{-1}[aF(s) + bG(s)] = a\mathcal{L}^{-1}[F(s)] + b\mathcal{L}^{-1}[G(s)]$$

where a and b are constants.

- Convolution Property

Given two Laplace transforms, $F(s)$ and $G(s)$, then

$$\mathcal{L}^{-1}[F(s)G(s)] = (f * g)(t)$$

General Strategy for Inverse Laplace Transform

Looking at the equation we left off on, there is no direct Laplace Transform to get y. The next step involves heavy involvement of partial fractions and decomposition. Generally, we want one of more terms that all have s in the denominator. The good thing is that the factor in the dominator follows different forms that has corresponding partial fraction decompositions, if you remember from Calculus II (I believe). For a refresher, look at the following chart posted on Paul's Online Notes:

Factor in denominator	**Term in partial fraction decomposition**
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$\left(ax^2+bx+c\right)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

After identifying what factor is in your denominator, you should set your term equal to the partial fraction decomposition. You would then want to select values that would select s values that isolate one of the As or B to get their values. Once you do this, you can do inverse Laplace transforms.

That was a simplification. Let's see this in practice.

Practice with Inverse Laplace Transform – Continuation

We left off with this:

$$\mathcal{L}[y]=\frac{sf(0)+f'(0)-f(0)}{s^2-s-2}$$

Suppose I got the initial conditions of $y(0)=1, y'(0)=0$. Let's plug those in:

$$\mathcal{L}[y]=\frac{s+0-1}{s^2-s-2}$$

$$\mathcal{L}[y] = \frac{s-1}{s^2-s-2}$$

In this case, I see an s time and notice that the denominator can be factored. Let's try that and see if we get any cancellations. If you are confused by factoring, familiarize yourself with the quadratic formula.

$$\mathcal{L}[y] = \frac{s-1}{s^2-s-2}$$

$$\mathcal{L}[y] = \frac{s-1}{(s-2)(s+1)}$$

Alright, that was a bust, no simplification can be done here. Alright, let's identify what the denominator looks like. There are two terms of s and a number. So, we know there are going to be two terms on the right-hand side of the equation. The fact that the denominator is factorable, this signals this fact and make me not select the $ax^2 + bx + c$ partial fraction decomposition, although it was originally in that form. It only works if the quadratic is not factorable with integers.

$$\mathcal{L}[y] = \frac{s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

Let's work on the RHS of the equation and make the denominators of A and B be the same as the s terms:

$$\mathcal{L}[y] = \frac{s-1}{(s-2)(s+1)} = \frac{A(s+1)}{(s-2)(s+1)} + \frac{B(s-2)}{(s+1)(s-2)}$$

$$\mathcal{L}[y] = s-1 = A(s+1) + B(s-2)$$

(since the denominators are all the same, you can just get rid of it)

Now, let's work on solving for A. To do that, we need to set the B term equal to 0, which means setting $s-2=0$. By equation manipulation, we find that s = 2. So, let's plug in 2!

$$2-1 = A(2+1) + B(2-2)$$

$$2-1 = A(2+1)$$

$$1 = 3A$$

$$A = \frac{1}{3}$$

Now that we know A, let's do the same thing to find B and make s = -1.

$$-1-1 = A(-1+1) + B(-1-2)$$

$$-2 = B(-1-2)$$

$$-2 = -3B$$

$$B = \frac{2}{3}$$

Perfect. Now, we plug A and B into this equation:

$$\mathcal{L}[y] = \frac{s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

$$\mathcal{L}[y] = \frac{s-1}{(s-2)(s+1)} = \frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1}$$

Now, we can take the inverse Laplace transform:

$$\mathcal{L}[y] = \frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1}$$

By the rules of linearity for inverse Laplace transforms, we get:

$$\mathcal{L}^{-1}[\mathcal{L}[y] = \frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1}]$$

$$y = \mathcal{L}^{-1}[\frac{\frac{1}{3}}{s-2}] + \mathcal{L}^{-1}[\frac{\frac{2}{3}}{s+1}]$$

$$y = \frac{1}{3}\mathcal{L}^{-1}[\frac{1}{s-2}] + \frac{2}{3}\mathcal{L}^{-1}[\frac{1}{s+1}]$$

Referring to the first chart, we get the following:

$$y = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

Awesome! We went through a full problem of solving a differential equation with Laplace transforms. For extra practice and to familiarize yourselves more with the topics I covered, refer to the following videos:

- Houston Math Prep
- blackpenredpen
- BriTheMathGuy (for Laplace transform practice with piecewise functions)

- Paul’s Online Notes
- LibreTexts Mathematics

First Translation Theorem

We have a new section so more definitions and theorems! The next section deals with operation functions which are properties and theorems used to transform complex differential equations in algebraic equations. Technically, I have already used a few so far such as linearity and convolutions. However, there’s another one that’s crucial for solving differential equations called the first translation theorem aka frequency shifting. Again, if you will take Electrical Circuits or Signals and Systems, make sure you save your notes on these.

The First Translation Theorem states:

$$\mathcal{L}[e^{at}f(t)] = F(s-a)$$

It is particularly helpful when you want to shift or translate your function in the t space to s space and vice versa. I think it’s a bit hard to describe without doing a problem, so let’s do that.

Let’s say I want to find the Laplace transform of t^3e^{-2t}.

According to the theorem, the shifting term is defined by the exponential, so let’s instead zero in on the t^3 term, which is the $f(t)$. Taking the Laplace transform of it, we use the following form (refer to the table above):

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[t^3] = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$$

Now that we have the basic form, we can account for our shifting term. According to the theorem, a = -2. This makes $s = s-(-2) = s+2$, or translating it to the left on the s-axis. Therefore, the Laplace transform of the full term is:

$$\mathcal{L}[e^{-2t}t^3] = \frac{6}{(s+2)^4}$$

You can also use it for the inverse Laplace transform. Let’s say we are given $F(s) = \frac{1}{s^2+12s+36}$. Let’s see if we can simplify the denominator. Based on the two terms to the right on the denominator, there is a factor of 6 that multiplied to itself equals 36 and the sum also equals 12. Therefore,

$$s^2 + 12s + 36 = (s+6)^2$$

This makes:

$$F(s) = \frac{1}{s^2 + 12s + 36} = \frac{1}{(s+6)^2}$$

We see that is we ignored the + 6, the form would just be:

$$\frac{1}{s^2}$$

Based on our knowledge of inverse Laplace transforms, this means that $f(t) = t$. Okay, know we need to account for the + 6 term we ignored. This is where we add this exponential. If

$$\mathcal{L}[e^{at} f(t)] = F(s-a)$$

Then, if the a in Laplace transformed function was positive, the a must be negative. Let's apply this to our $f(t)$.

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 + 12s + 36}\right] = te^{-6t}$$

Hopefully this is making sense to y'all. If not, here are some others who explain the topic and do practice problems:

- EasyMath
- Jonathan Walter
- Dr. Chis Tisdell
- LibreTexts Mathematics

That's all I have for Laplace transforms, generally! If you are also expected to know the other theorems that I did not explicitly go through here, refer to the following resources:

- Rutgers Handout
- EasyMath playlist
- Paul's Online Notes

I hope this helps y'all in studying and introducing concepts! If you are still shaking on things, feel free to come in during the Math hours!