MATH 2110Q – Exam 3 Review

Hello everyone, it's Akeva! This document is going to cover the topics that you will be tested on in Multi for Exam 3 (as of Spring 2026): Vectors, Arc Length, Line Integrals, Vector Fields, Green's Theorem, Curl and Divergence, Parametric Surfaces and Areas, and Surface Integrals. I know it seems like a lot but many of these topics are related and build upon past concepts. I will take things section by section and give my tips and tricks to solving these problems as usual!

As usual, underlined words are headers for new sections and italicized words are subsections. Definitions will be labeled in red and problems I do will be done in purple.

Vector Functions

What is a vector function or vector-valued function?

It's a function whose domain is a set of real numbers and whose range is a set of vectors. You can think of this it as inputting putting numbers (any type of number from rational to irrational) and getting an associated vector (remember, this means it has magnitude and direction). For the sake of Multi, we focus mainly on functions that produce three-dimensional vectors.

The vector would be defined as:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

Where f, g, and h are called component functions of r.

Finding Limits and Continuity

Just as in previous sections and other finding the limits and determining whether function is continuous is important for future applications. So, how do we find the limits of a vector function? It's defined as follows:

$$\lim_{t\to a} \mathbf{r}(t) = \langle \lim_{t\to a} f(t)\, , \lim_{t\to a} g(t)\, , \lim_{t\to a} g(t)\ \rangle$$

If the inner limits exist, then the vector function itself exists as that point. Furthermore, if the limit of the vector function exists, the vector function is continuous at that point.

Let's do a quick problem together on this concept. I will walk through my thought process on this. This problem is derived from Paul's Online Notes. Let's compute the following:

$$\lim_{t\to 1} \langle t^3, \frac{\sin(3t-3)}{t-1}, e^{2t}\rangle$$

I see that the first term and second terms do not have a t variable in the denominator, so I can just plug and chug there. The first term will simply be 1 and the third term would be e^2.

Now, for the second term, since there is a t in the denominator and plugging in , using L'Hôpital's rule, we should automatically take the derivative of the top and bottom. This gets us:

$$\frac{dr}{dt}\left(\frac{\sin(3t-3)}{t-1}\right) = \frac{-3\cos(3t-3)}{-1} = 3\cos\,(3t-3)$$

Plugging in 1, we get:

$$3\cos(3*1-3) = 3\cos(3-3) = 3\cos(0) = 3(1) = 3$$

Therefore, the limit is,

$$1\mathbf{i} + 3\mathbf{j} + e^2\mathbf{k}$$

Hopefully that all made sense! If not, try out some more problems will the links below:

- Krisa King
- Jonathan Walters

Space Curves

If we focus on all points that can be created in an interval and f, g, and h are continuous real-valued functions on that interval, those set of points, C, are called a space curve. This works when the following is true:

$$x = f(t), y = g(t), and\ z = h(t)$$

The above equations are referred to as the parametric equations of C and t is the parameter.

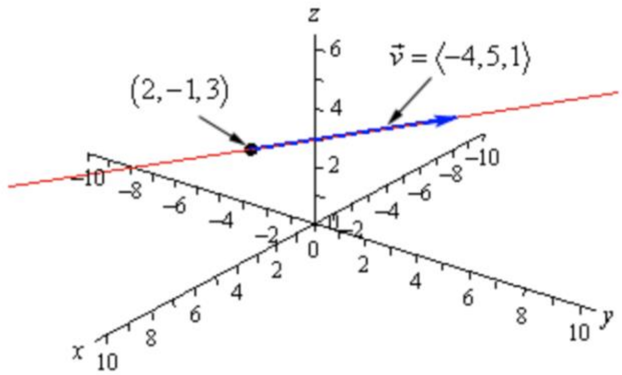
Based on these parametric equations, you should be able to recognize the overall space curve or draw it. I will do out two problems from Paul's Online Notes. Because I am only working on a computer, I won't draw out the final graph but will include my thinking for what the overall shape will look like and provide the sketch from the website. First let's think about shape of the following vector function:

$$\mathbf{r}(t) = \langle 2-4t, -1+5t, 3+t\rangle$$

When I approach this, I think about trying to simplify the vector by grouping common terms or into something familiar. I see that each component of the vector has a number and then a term with the parameter t. So, group everything together:

$$\mathbf{r}(t) = \langle 2, -1, 3\rangle + \langle -4, 5, 1\rangle t$$

If we think back to a prior unit or even think back to Algebra, this looks just like an equation for a line. It's a line that goes through the point (2,-1,3) and is parallel to the vector $\langle -4,5,1\rangle$. Awesome! Now what's left is to just plot everything. Below is what a sketch looks like.



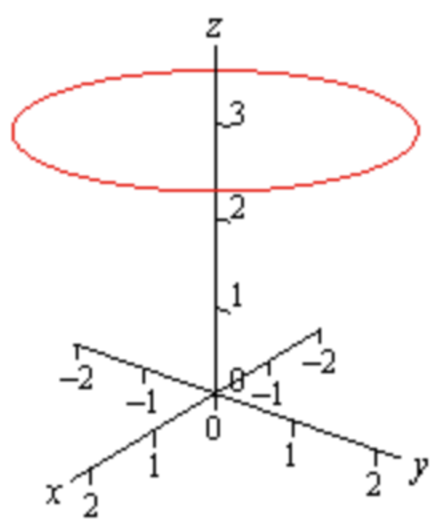
Let’s tackle a second problem. Let’s think about the following vector function:

$$\mathbf{r}(t) = 2\cos(t)\,\mathbf{i} + 2\sin(t)\,\mathbf{j} + 3\mathbf{k}$$

My eyes are automatically drawn to the cosine and sine functions, making me think of a circle. Looking at the third term, I see that it’s not dependent on t, meaning it’s fixed. This makes me see that it’s a circle. However, let’s group like terms as we did last time:

$$\mathbf{r}(t) = 2(\cos(t)\,\mathbf{i} + \sin(t)\,\mathbf{j}) + 3\mathbf{k}$$

Okay, so we see that we are deal with a circle on the xy plane about the z-axis at 3, as opposed to 0. Try sketching this before you refer to the sketch below!



There is another type of problem that you should be able to do, and that is determining parametric equations for an intersection of a plane and a shape. I will do one problem out from the textbook (Example 6 in Chapter 13).

The prompt asks us to "find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$."

Based on the shape and the plane, we can infer that the intersection will be circular and elliptical in shape because the plane cuts through cylinder in half. Therefore, our x and y components of the vector will have cosine and sin respectively and t should be bound between 0 and 2pi. There are also no extra constants since there are none that appears in the cylinder equation. Now, we need to determine the z component. Since, we know y, which is sin (t), we can plug that into the plane equation:

$$\sin(t) + z = 2$$

$$z = 2 - \sin(t)$$

Now, that we have our z, we can combine everything together and get the vector equation:

$$\mathbf{r}(t) = \cos(t)\,\mathbf{i} + \sin(t)\,\mathbf{j} + (2 - \sin(t))\mathbf{k}$$

Since there is not only a constant in the z components, we know that the intersection is elliptical, not circular.

If you need more problems to do to practice these, use the following resources:

- Grantstrong (sketching)
- Krista King (sketching)
- Krista King (intersection)
- Jonathan Walters (intersection)

Derivatives and Integrals

The derivative for a vector function is useful for determining the slope at a given point, just as it was in Calc I. Given that f, g, and h and differentiable, then the derivative of the vector function is:

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

As a note, the second derivative is calculated in a similar fashion: by components. A key concept dependent on the derivatives of vector functions is the unit tangent vector. It is defined as:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

The unit tangent vector is in the same direction as the tangent vector, but the magnitude is overall 1. As a refresher, the magnitude of a vector is defined as $\sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2 + \left(h'(t)\right)^2}$.

There are theorems that are particularly useful to for differentiating vector functions. Let's assume u and v are differentiable vector functions, c is a scalar, and f is a real-value function. Then

1. $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\frac{d}{dt}[\mathbf{u}(t) * \mathbf{v}(t)] = \mathbf{u}'(t) * \mathbf{v}(t) + \mathbf{u}(t) * \mathbf{v}'(t)$
5. $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

For practice with finding the tangent vector, use the following resources:

- CBlissMath
- Krista King

With regards to integrals, a definite integral of a vector function is defined as:

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}$$

For practice with finding definite integrals of vector functions, use the following resources:

- Krista King
- JK Math (46 mins)
- Math with Professor V (32 mins)

Arc Length

This next section deals with this thing called arc length, defined as L. It essentially is the plane curve made of approximated polygonal paths. Instead of a smooth curve, it would look like distinct lines. If you draw, think about how often you start with straight lines before introducing curves into your work. The arc length essentially connects two points on the curve with a straight line. The equation for this is below:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = |\mathbf{r}'(t)|dt$$

There is also an associated function to arc length which parameterizes a curve with respect to arc length. You may ask why we need an arc length function. It's useful because length comes from the shape of the curve and does not depend on the coordinate system. A typical problem would ask you to reparametrize a vector function, which should signal to you that you should use the function defined below:

$$s(t) = \int_a^t |\mathbf{r}'(u)|\, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$

With regards to the reparameterization problems, you should calculate the magnitude of the unit vector first and then calculate the value of s. Based on the value, you can determine t and substitute the parameter t with the calculated s in the vector function.

For examples of these types of problems and practice, looking at the following:

- Arc length
  - Krista King
  - Dr. Trefor Bazett (concept overview)
  - JK Math (25 mins)
- Reparamertization
  - Krista King
  - Dr. Bevin Maultsby

Vector Calculus

Vector Fields

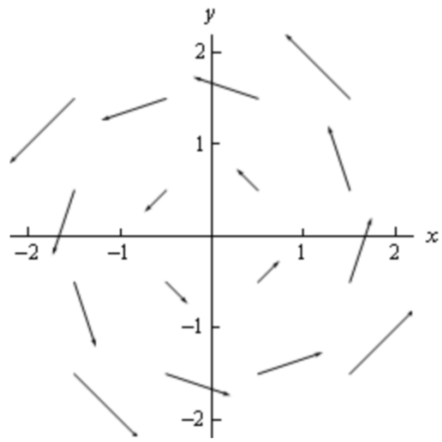
Extending from vector functions, we have these things called vector fields which are arrows that stem from the point on the coordinate and points in a direction defined by component functions P and Q in $\mathbb{R}^2$ and P, Q, and R in $\mathbb{R}^3$ and make up a vector $\mathbf{F}$. $\mathbf{F}$ is defined as the following in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively:

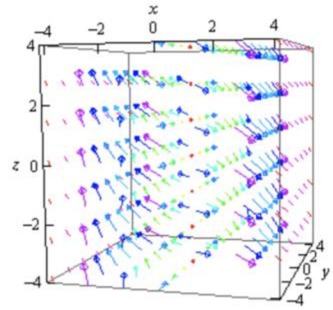
$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x.y)\mathbf{j} = \langle P(x, y), Q(x, y)\rangle = P\mathbf{i} + Q\mathbf{j}$$

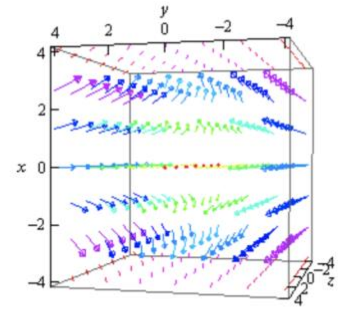
$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

Based on the vector fields, you can determine the overall shape of $\mathbf{F}$.

Examples of vector fields for $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$ and $\mathbf{F}(x, y, z) = 2x\mathbf{i} - 2y\mathbf{j} - 2x\mathbf{k}$ can be found below (left to right):







Line Integrals

Line integrals are like an ordinary integral but instead of integrating over the interval [a,b], we instead integrate over a curve, C. You may be confused because in a normal integral, we take the integral to find the area under a function. You can think of a line integral as the 3D version of an integral where we calculate the total effect of a field (whether that be a vector or scalar) on a curve. With regards to an equation, if f (any function of two variable whose domain includes the curve C) is defined on a smooth curve C, then the line integral of *f* along C is the following if it exists:

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Oftentimes, it is easier to solve these problems when you do parameterization aka defining everything in terms of t as opposed to x and y, especially when the bound of the curve is not defined. This step requires remembering the conversions and writing common shapes with parametrizations. Let's do a couple of problems so I can explain my approach. If you need a basic review of parametric equations for circles and ellipses, refer to the table linked here by Paul's Online Notes. I am also doing a problem linked in the same page. It's as follows:

Evaluate $\int_C xy^4 ds$ where C is the right half of the circle $x^2 + y^2 = 16$ traced out in a counterclockwise direction.

For this problem, C is not defined in terms of points, therefore we need to parameterization. Based on the defined curve and an explicit mention, we know the curve is a circle. Looking at the right-hand side (RHS) of the equation, we see that the radius is 4 (the square root of RHS). So,

$$x = 4\cos(t) \quad y = 4\sin(t)$$

Considering that we are focusing on the right half of the circle, and going counterclockwise, we know the bounds should be from -$\frac{\pi}{2}$ to $\frac{\pi}{2}$. Now we can plug this into the given equation:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos(t))(4\sin(t))^4 \sqrt{(-4\sin(t))^2 + (4\cos(t))^2}\,dt$$

That looks like a lot, let's simplify:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos(t))(4\sin(t))^4 \sqrt{16\sin^2(t) + 16\cos^2(t)}\,dt$$

Based on the trig identity, $\sin^2(t) + \cos^2(t) = 1$,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos(t))(4\sin(t))^4 \sqrt{16}\,dt$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos(t))(4\sin(t))^4 (4)dt$$

$$4096 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t)\sin^4(t)\,dt$$

That's better. But we need to further simplify the integral. Based on chain rule, we can see the cosine term being produced by the 5$^{\text{th}}$ power of sine of t (if you're confused, take the derivative of the term below in the brackets; you should end up with the above term):

$$4096 \left| \frac{\sin^5(t)}{5} \right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$\frac{4096}{5}|\sin^5(t)|\,\begin{matrix}\frac{\pi}{2}\\ -\frac{\pi}{2}\end{matrix}$$

$$\frac{4096}{5}[\left(\sin\frac{\pi}{2}\right)^5-\left(\sin-\frac{\pi}{2}\right)^5]$$

$$\frac{4096}{5}[(1)^5-(-1)^5]$$

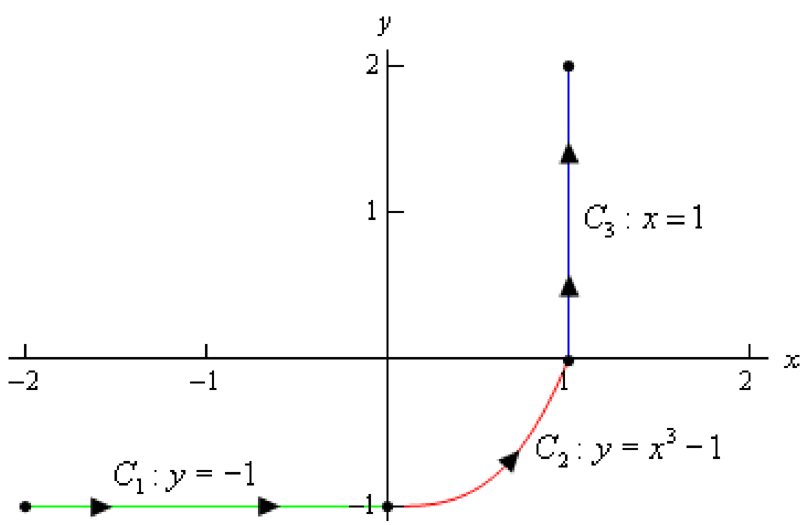
$$\frac{4096}{5}[1-(-1)]$$

$$\frac{4096}{5}[2]$$

$$\frac{8192}{5}$$

I'll also do a problem where C is a distinct curve with points, so you know how to do the general two types of problems with line integrals. This problem also comes the page I linked above for the first problem. The second problem is the following:

Evaluate $\int_C\ 4x^3ds$ where C is the curve shown below:



Looking at the curve, there are three distinct regions, so our line integral will be split up into three sections. Let's define x in terms of t and find the bounds for each section.

For C_1, since it's just $y = -1$, the x can be defined as t and the values for the bounds will be -2 and 0 (following the x-axis above).

For C_2, x can once again be just t while $y = t^3 - 1$. The points defined for C_2 is (0,-1) and (1,0). Since we are focusing on the values of t, which is just x, the bounds are 0 and 1.

For C_3, things are a little bit different. x is no longer increases, in fact, it's just 1. However, y is increasing. Therefore, we set y equal to t. Following the curve and from where we left off, t is bounded from 0 to 2 (the bounds do not need to necessarily increases sequentially, especially when we change the variable we define as t).

So, what does this look like all together?

$$C_1: \int_{-2}^{0} 4t^3\sqrt{(1)^2 + 0^2}dt = \int_{-2}^{0} 4t^3 dt = t^4|_{-2}^{0} = (0^4 - (-2)^4) = -16$$

$$C_2: \int_{0}^{1} 4t^3\sqrt{(1)^2 + (3t^2)^2}dt = \int_{0}^{1} 4t^3\sqrt{9t^4 + 1}\, dt = \int_{0}^{1} 4t^3(9t^4 + 1)^{\frac{1}{2}}$$

For C_2, we can't simplify further. If we look though, what's under the square root, excluding constants, is the term outside the square root. This means we have to use u-substitution.

$$u = 9t^4 + 1$$

$$du = 36t^3$$

$$t^3 = \frac{du}{36}$$

$$\int_{0}^{1} 4t^3\sqrt{9t^4 + 1}\, dt = 4\int \frac{\sqrt{u}}{36} du = \frac{1}{9}\left(\frac{2}{3}u^{\frac{3}{2}}\right) = \frac{2}{27}(9t^4 + 1)^{\frac{3}{2}}\Big|_0^1 = \frac{2}{27}((10)^{\frac{3}{2}} - 1)$$

Let's move on to C_3 now:

$$C_2: \int_{0}^{2} 4\sqrt{(0)^2 + (1)^2}dt = \int_{0}^{2} 4\, dt = 4t|_0^2 = 8 - 0 = 8$$

Combining everything, we get:

$$-16 + \frac{2}{27}\left(10^{\frac{3}{2}} - 1\right) + 8$$

Awesome, we completed two problems of line integrals in a plane. Let's look at the equations of two other types of integrals.

There are line integrals with respect to x or y or line integrals with respect to arc length and are defined as:

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\;x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\;y'(t)\,dt$$

When asked to parametrize a line segment, use the following equation:

$$\mathbf{r}(t) = (1-t)\mathbf{r_0} + t\mathbf{r_1} \quad 0 \le t \le 1$$

Where r_0 is the start of the segment and r_1 is the end.

Furthermore, you can also have line integrals in 3D. Their form takes the following:

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

Or more compactly in vector form,

$$\int_a^b f\big(\mathbf{r}(t)\big)\,|\mathbf{r}'(t)|\,dt$$

If you want more practice and/or need more explanation, refer to the following resources:

- Dr. Trefor Bazett (what is a line integral)
- Dr. Trefor Bazett (example)
- Alexandra Niedden (45 mins)
- Professor Leonard (~2 hrs)
- Washington University Practice and Solutions (worksheet)
- James Hamblin (line segment)
- Krista King (vector form)

Fundamental Theorem for Line Integrals

The Fundamental Theorem for Line Integrals stems from the idea that the gradient vector ∇f of a function f of two or three variables can be regarded as a derivative of *f*. Therefore, if C is a smooth curve given by the vector function **r**(t), where $a \le t \le b$, and *f* is differentiable whose gradient vector is continuous on C, then:

$$\int_C \nabla f * \mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

If f is a function of two variables and C is a plan curve with an initial point $A(x_1, y_1)$ and terminal point $B(x_2, y_2)$, then the theorem becomes

$$\int_C \nabla f * \mathrm{d}\mathbf{r} = f(x_2, y_2) - f(x_1, y_1)$$

This can be also can be extended to three variables.

Furthermore, if the C_1 and C_2 are smooth curves with the same initial points and terminal points, we know they equal each other:

$$\int_{C_1} \nabla f * \mathrm{d}\mathbf{r} = \int_{C_2} \nabla f * \mathrm{d}\mathbf{r}$$

Just as a reminder, the gradient vector is defined as the following for two variables and three variables, respectively:

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$$

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$$

Some common question you may be asked in relation to the Fundamental Theorem for Line Integrals are determining if paths are independent and determining if a field is conservative. To solve these problems, let's define the underlying concepts in each:

- Independence of Paths
    - For a continuous vector field, F, with a domain D, the line integral is independent of path if $\int_{C_1} F * dr = \int_{C_2} F * dr$ for any two paths C_1 and C_2 in D that have the same initial and terminal points.
    - Therefore, $\int_C F * dr$ is independent of path in D if and only if $\int_C F * dr = 0$ for every closed path C in D
        - A closed path is when the terminal point equals the initial point
- Conservative Vector Fields
    - As an extension, if the vector field is continuous on an open connected region R and $\int_C F * dr$ is independent of path in D, then F is a conservative vector field on D, meaning there exists a function *f* where $\nabla f = \mathrm{F}$.

        - Open region is one that does not contain its boundary points
    - To determine if a vector field is conservative, you would use the following theorem:
        - If $F(x, y) = P(x, y)i + Q(x, y)j$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

To keep this document from being too long, I will link some resources below that helped me better understand the theorems I went through. Feel free to come in though, if you have more questions!

Resources:

- Dr. Bevin Maultsby (Fundamental Theorem)
- Patrick J (Fundamental Theorem)
- Krista King (Independence of path)
- LibreTexts Mathematics (Independence, Conservative Fields)

Green's Theorem

Alright, new section, so we need some more definitions. Green's Theorem is often described as the relationship between a line integral, which is integrating over a simple closed curve (aka no weird closures are empty space in the middle), and a double integral over the plane region bounded by the curve. I like to think of it as a way to calculate 2D area of a curve that's not necessarily easy to compute (like a circle or a square).

When using Green's Theorem, there is a typical way to travel along the curve to calculate the area called the positive orientation. This direction is simply a single counterclockwise traversal of the curve.

The official definition of Green's Theorem is if C is a positively oriented, piecewise-smooth, simple closed curve in the plane, D is the region bounded by C, and P and P and Q have continuous partial derivatives on an open region that contains D, then:

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

The notion for this can also be represented as the following:

$$\oint_C P\,dx + Q\,dy$$

To use the Green's Theorem to compute the area, the reverse direction should be used and gives us the following equation:

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$$

To practice with Green's Theorem, use the following videos and resources as guidance:

- Professor Dave Explains
- Professor Leonard (1 hr and 45 mins)
    - Goes through problems and concepts
- Patrick J
- Math with Professor V (41 minutes)
- Paul's Online Notes

Curl and Divergence

In this section, you will learn about curl which may just seem like another concept. However, if you need to take Physics or have already taken it, these concepts will not be new to you.

Let's start with curl. A curl of a vector field, which is denoted as ∇ x F, is a magnitude and direction of a "swirling" tendency of a 3D field at a point. You may remember in Physics II, you used the Right Hand Rule to determine the direction of a magnetic field around a wire. This just gives a magnitude to the field, if you will, for a particular point in a vector field. How do we express this in a mathematical form?

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, where **F** is a vector field and P, Q, and R are partial derivatives and they exist, then the curl of F is defined as:

$$\text{curl}\ \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) i + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) k$$

One useful thing to think about is that if the f is a function of three variables that has continuous second-order partial derivatives, then

$$curl(\nabla f) = 0$$

Furthermore, if **F** is conservative, then curl **F** is zero.

Now, let's focus our attention on divergence, the second topic of this section. Divergence is the flux of a vector field at a certain point; therefore, it is a number. A negative value represents a sink where the vector field is coming to that point while a positive value means that point is a source where the vector field is stemming from that point. Zero means that the flow in and out is equal. The formal definition of divergence, also denoted as $\nabla \cdot \mathbf{F}$, is:

$$\text{div}\ \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

If F is a 3D vector field, and P, Q, and R have continuous second-order partial derivatives (aka you do the partial derivative twice), then:

$$\text{div curl}\ \mathbf{F} = 0$$

With these definitions, we could also rewrite the equation in Green's Theorem in vector form such that:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_D (curl\ \mathbf{F}) \cdot \mathbf{k}\, dA$$

Where F is the vector F, T is the unit tangent vector, and k is the third component. This can be particularly useful is it's easier to calculate the curl or have already done so for a problem.

For extra practice with curl and divergence, please use the following resources:

- Professor Dave Explains
- Krista King
- Houston Math Prep
- Steven Bruton (explanation of concepts)

Parametric Surfaces and Their Areas

So far y'all have worked with many surfaces: cylinders, quadratic surfaces, spheres, cylinders, graphs with two and three variables, and level surfaces. But there is still a class that we haven't gone over yet called parametric equations. They are very similar yet the way in which they are defined is different and a particularly useful to model curves or paths that fail the vertical line test (remember that if something has many y values for a single x, it fails the vertical line test and is therefore not a function). Instead of a single relationship, we have the parametric equations:

$$x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

Note that this isn't the first time we have used parametric equations but now you know the name for them. Generally, these variables are connected by the following function:

$$r(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

The parametric refers to u and v in this case as they serve as our parameters to the vector function. The set of all points (x,y,z) in the 3D space that following the parametric equations above and (u,v) varies through space, D, is called a parametric surface.

Let's do a problem together where we think about identifying and drawing parametric equations. The following is from your textbook:

Identify and sketch the surface with vector equation

$$r(u,v) = 2\cos u\,\mathbf{i} + v\,\mathbf{j} + 2\sin u\,\mathbf{k}$$

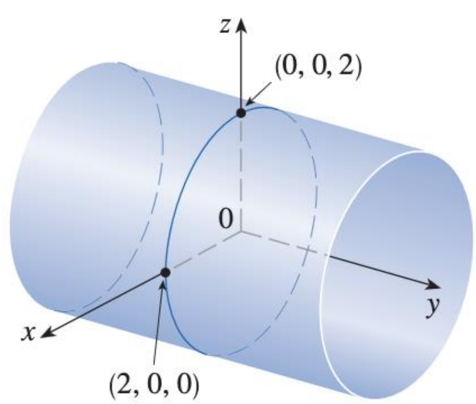
Okay, let's identify x, y, and z.

$$x = 2\cos u$$

$$y = v$$

$$z = 2\sin u$$

I automatically look at the sine and cosine functions in x and z. This signals to me circle with a radius of 2. Looking at y, it's not fixed so we just see that multiple circles of the same size moves up and down the y-axis. Putting all of this together, we get a cylinder fixed on the xz plane. It should look this:



Another common question to be asked is finding a vector function. When it comes to identifying points, you should refer to my previous guide where I touch upon finding a vector equation using two points. For other complex shapes, you have to take different approaches. Let's do a problem together (from one of Dr. Bevin Maultsby's videos on YouTube):

"Parameterize the upper hemisphere $z = \sqrt{4 - x^2 - y^2}$."

Let's try doing this without polar coordinates. Since z is already defined in terms of x and y, let's just make x equal to u and y equal to v. Substituting that into z, we get $z = \sqrt{4 - u^2 - v^2}$.

Looking at the equation, we see a $u^2 + v^2$ term, which signals to be we have a circle. But based on the 4, we know that the circle is limited to a radius of 2. Okay, but when we parametrize, we also need to know our bounds (will come handy later).

So, let's solve for one of the variables. I will do it for v, since in my mind that's like solving for y (we get the below equation by setting z equal to 0 and rearranging):

$$u^2 + v^2 = 4$$

$$v^2 = 4 - u^2$$

$$v = \pm\sqrt{4 - u^2}$$

This means that v is bounded by the positive and negative of the square root. Okay, what about u, what are the bounds for it? Well, we know that everything under square root but be greater than or equal to 0. So, let's work with that:

$$4 - u^2 = 0$$

$$4 = u^2$$

$$u = \pm 2$$

Therefore, u is bounded between -2 and 2. Try doing the same thing but parameterizing with polar coordinates. For guided help, use the video above.

For more practice sketching parametric curves and finding vector functions, use the following resources:

- Paul's Online Notes (sketching)
- Dr. Bevin Maultsby (finding parameters)
- Meredith Burr (finding equations)
- LibreTexts Mathematics (sketching, finding parameters)

Parametric equations can also present key shapes that we have seen before in this course: surfaces of revolution and tangent planes. With regards to the former, it is based on the following parametric equations:

$$x = x \qquad y = f(x)\cos\theta \qquad z = f(x)\sin\theta$$

$f(x)$ here is defined as the curve, which is rotated above the x-axis. Tangent planes of a parametric equation can be defined by the following equations:

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_o, v_o)\mathbf{i} + \frac{\partial y}{\partial v}(u_o, v_o)\mathbf{j} + \frac{\partial z}{\partial v}(u_o, v_o)\mathbf{k}$$

$$r_u = \frac{\partial x}{\partial u}(u_o, v_o)\mathbf{i} + \frac{\partial y}{\partial u}(u_o, v_o)\mathbf{j} + \frac{\partial z}{\partial u}(u_o, v_o)\mathbf{k}$$

The first equation is obtained when we keep u constant and the second equation is when we keep v constant at point P_o. If $r_u \times r_v$ is never 0, then the surface S is smooth. Furthermore, the cross product of the two tangent vectors is a normal vector (90 degrees) to the tangent plane. As a reminder of how to get the tangent vector from here, you would be given a point and simply shift the x, y, and z values by the point. For example, let's say we calculated the tangent plane to be $-4vi + 3uj + 7uvk$ for the point (2, 8, 3) (assuming u and v both equal 1, if not, you should plug that in before doing the next step). Then, the equation for the tangent plane is:

$$-4(x-2) + 3(y-8) + 7(z-3) = 0$$

There is a final helpful function parametric equations yield: they can help calculate the surface area of a smooth parametric surface. If a smooth parametric surface, S is given by the following equation:

$$r(u,v) = x(u,v)i + y(u,v)j + z(u,v)k \qquad (u,v) \in D$$

And S is covered just once as (u, v) ranges throughout the parameter domain D, then the surface area of S is

$$A(S) = \iint_D |r_u \times r_v| \, dA$$

For a surface S with equation $z = f(x,y)$, where (x,y) lies in D and f has continuous partial derivatives, we take x and y as parameters, making the parametric equations:

$$x = x \quad y = y \quad z = f(x,y)$$

making

$$r_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}\right)\mathbf{k} \qquad r_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}\right)\mathbf{k}$$

The normal vector then turns into

$$|r_x \times r_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}$$

Plugging this fact into the general equation for surface area above, we get the following:

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA$$

For practice with tangent planes and surface area for parametric equations, use the following resources:

- Tangent Plane
  - Michael Hutchings
  - Krista King
  - Brenda Edmonds
- Area of Surfaces
  - JK Math
  - Krista King
  - Paul's Online Notes (Examples 3 and 4)

Surface Integrals

We are almost there; this is the last section. If you haven't already, take a break and then come back so we can tackle this.

I hope you did take a small break. Now, let's get into surface integrals. You may be wondering what the difference is between surface area and surface integrals. Think of surface area as something that tells us solely about the surface (aka only geometry), while surface integrals tell us a cumulative value of a function over a given surface (aka geometry and behavior of a field on the surface). Therefore, you would use a surface integral when you want to calculate the electrical charge enclosed by a surface or total rate of heat passing through a surface. Note the difference between surface integrals and curl and divergence. They focus on points as opposed a surface. The equation for a surface integral is the following:

$$\iint_S f(x, y, z)\, dS = \iint_D f\big(r(u, v)\big)|r_u \times r_v|\, dA$$

where the left-hand side of the equation is defined as the surface integral of f over the surface S. I also want to note that when $f(x, y, z) = 1$, then you simply get the surface area.

For any surface S with equation $z = g(x, y)$, it can be regarded as a parametric surface with parametric equations:

$$x = x \quad y = y \quad z = g(x, y)$$

making

$$r_x = \mathbf{i} + \left(\frac{\partial g}{\partial x}\right)\mathbf{k} \qquad r_y = \mathbf{j} + \left(\frac{\partial g}{\partial y}\right)\mathbf{k}$$

And

$$|r_x \times r_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

Substituting the cross product in the equation for a surface integral, we get the following:

$$\iint_S f(x,y,z)\, dS = \iint_D f(x,y,g(x,y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\, dA}$$

You can easily change this equation when projecting S onto another plane. The above equation is for the xy plane, but, for example, the projection onto the yz plane:

$$\iint_S f(x,y,z)\, dS = \iint_D f(i(y,z),y,z) \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2}\, dA$$

Considering the fact that surface integrals in a way intersect with a surface and a function, the manner in which we approach a surface can greatly influence the result of a surface. For example, if want to determine heat flux through an object, the perspective of facing the inside of a surface or looking outside the surface, can influence the value. We call this an oriental surface. We can look at the direction by looking at the normal vector which can be calculated in the following way:

$$\mathbf{n} = \frac{r_u \times r_v}{|r_u \times r_v|}$$

If k-component is positive, one can assume that the normal vector is in the upward orientation (pointing away the surface). If it is negative, that means it is facing inside the surface.

With regards to flux, if F is a continuous vector field defined on an oriented surface S with unit normal vector n, then the surface integral of **F** over **S** or the flux of **F** across **S** is

$$\iint_S \mathbf{F} \cdot d\boldsymbol{S} = \iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$$

If S is given by a vector function $r(u,v)$, then we get:

$$\iint_S F \cdot dS = \iint_D F \cdot (r_u \times r_v)\, dA$$

Furthermore, if a surface S is given by a graph = $z = g(x,y)$, we can think of x and y as parameters making the equation take the following form:

$$\iint_S F \cdot dS = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

For practice/further explanations with flux, oriented surfaces, and surface integrals overall, use the following resources:

- Dr. Brevin Maultsby (Flux)
- Houston Math Prep (Flux)
- Dr. Brevin Maultsby (Parametric surfaces)
- Brian Mulholland (Oriented Surfaces)
- LibreTexts Mathematics (Oriented Surfaces)
- Math with Professor V (Surface Integrals and Flux)
- Paul's Online Notes (Surface Integrals)
    - Additional problems

That's all I have for an Exam 3 review! I know this was a longer document, but I wanted to cover almost everything that y'all will be responsible for. I hope this was helpful! Good luck studying!