University of Connecticut
Department of Mathematics

MATH1132Q | EXAM 3 PRACTICE | FALL 2024

NAME: Solutions

SIGNATURE: ______________________

TA Name: ______________________ Class Time: __________

Read This First!

- Please read each question carefully. The point value of each question is indicated in the box preceding its statement.
- In all *short answer* questions, you will *only* receive credit by providing clear work using techniques learned in this class.
- No books, formula sheets (other than the one provided), calculators, smart watches, or other references are permitted.
- No personal scrap paper is allowed. If you need additional space to write, it will be provided by your instructor. All scrap paper must be turned in with the exam.

Notes about the Practice Exam

- This practice exam is about twice as long as the actual exam.
- Please use this as a practice exam and try each problem before jumping to the solutions.
- The format of the exam will be similar to this practice exam, but you should not expect the same problems (or even the same kinds of problems).
- Using this as a form of studying is encouraged, but you are expected to study beyond this as well using your notes, homework, extra practice problems, office hours, study groups, etc.

Formula Sheet

Taylor's Inequality

Taylor's inequality on the interval $|x-a| \leq d$ is

$$|R_n(x)| = |f(x) - T_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \leq \frac{Md^{n+1}}{(n+1)!},$$

where M is chosen so that $|f^{(n+1)}(x)| \leq M$ for all x with $|x-a| \leq d$.

1) Indicate the validity of each statement by entering True (T) or False (F) in the blank.

(a) T If the Maclaurin series for $f(x)$ begins as $8+6x+3x^2+\cdots$ then $f''(0)=6$.

$\frac{f''(0)}{2!}x^2 \Rightarrow \frac{f''(0)}{2}=3$

$\Rightarrow f''(0)=6$

(b) T The center of the power series $\sum_{n=0}^{\infty} 2^n(x+4)^n$ is -4.

$(x-(-4))^n$

(c) T The radius of convergence of $\sum_{n=0}^{\infty} n!x^n$ is 0.

(Note: a power series always converges at it's center)

For $x\neq 0$: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right| = |x|\lim_{n\to\infty}(n+1) = |x|\cdot\infty < 1$ NEVER $R=0$

(d) F The radius of convergence for the Maclaurin series of $\frac{1}{1-x}$ is ∞.

$\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n$, I.O.C. : $|x|<1$

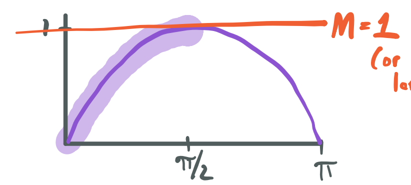
R.O.C. $=1$

(e) F If Taylor's inequality for $|f(x)-T_2(x)|$ when $f(x)=\cos(x)$ and $|x|<\frac{\pi}{2}$ we can use $M=\frac{1}{2}$.

$f'(x)=-\sin(x)$
$f''(x)=-\cos(x)$
$f'''(x)=\sin(x)$

$M \geq |f'''(x)| = |\sin(x)|$

on $|x|<\frac{\pi}{2}$, $|\sin(x)|<1$



(or anything larger than 1)

(f) T The differential equation $\frac{dy}{dx}=\frac{4}{xy+x}$ is separable.

$\frac{dy}{dx}=\frac{4}{x(y+1)} \Rightarrow (y+1)\,dy=\frac{4}{x}dx$

(g) F The orthogonal trajectories of $y=kx$ for constant k satisfy $\frac{dy}{dx}=\frac{y}{x}$.

$y=kx \Rightarrow \frac{dy}{dx}=k \Rightarrow \frac{dy}{dx}=\frac{y}{x}$ $\quad k=\frac{y}{x}$

negative reciprocal

orthog. trajectories

$\boxed{\frac{dy}{dx}=\frac{-x}{y}}$

(h) T The differential equation $\frac{dy}{dt}=y$ has a constant solution.

General sol'n $=$ $y=Ce^t$

so $y=0$ is a constant sol'n.

or

$\frac{dy}{dt}=\boxed{0=y}$

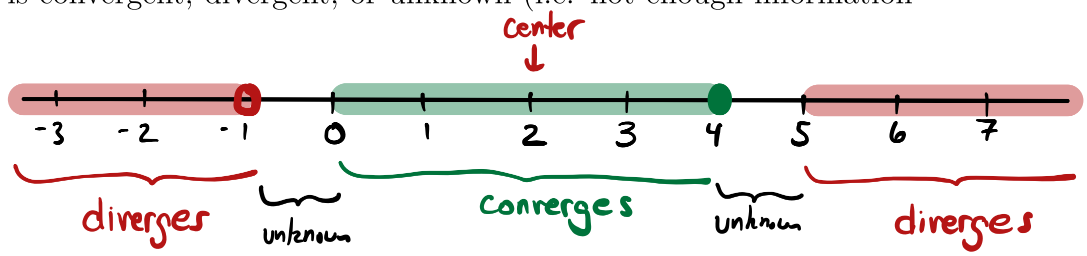
equilibrium solution

2) Find the power series representation for $f(x) = \dfrac{x^2}{4 - x^5}$.

Use $\dfrac{1}{1-\square} = \sum_{n=0}^{\infty} (\square)^n$

$$\frac{x^2}{4-x^5} = x^2 \frac{1}{4\left(1-\frac{x^5}{4}\right)} = \frac{x^2}{4}\sum_{n=0}^{\infty}\left(\frac{x^5}{4}\right)^n = \frac{x^2}{4}\sum_{n=0}^{\infty}\frac{x^{5n}}{4^n} = \boxed{\sum_{n=0}^{\infty}\frac{x^{5n+2}}{4^{n+1}}}$$

3) Consider a general power series centered at $x = 2$, $\sum_{n=0}^{\infty} c_n(x-2)^n$. Suppose that the series converges when $x = 4$ and diverges when $x = -1$. For each of the following series, indicate whether if it is convergent, divergent, or unknown (i.e. not enough information has been given).



(a) $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} c_n 1^n$

$= \sum_{n=0}^{\infty} c_n(3-2)^n$

$x=3$ $\boxed{\text{converges}}$

(b) $\sum_{n=0}^{\infty}(-1)^n 4^n c_n = \sum_{n=0}^{\infty} c_n(-4)^n$

$= \sum_{n=0}^{\infty} c_n(-2-2)^n$

$x=-2$ $\boxed{\text{diverges}}$

(c) $\sum_{n=0}^{\infty} c_n 3^n$

$= \sum_{n=0}^{\infty} c_n(5-2)^n$

$x=5$ $\boxed{\text{unknown}}$

Could diverge if $R<3$ or $R=3$ and I.O.C. $=(-1,5)$

Could converge if $R=3$ and I.O.C $=(-1,5]$.

4) Find the radius of convergence and interval of convergence of the following power series. (Show all work)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 2^n (x-3)^n}{n^2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}2^{n+1}(x-3)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(-1)^n 2^n (x-3)^n}\right| = \lim_{n\to\infty}\frac{2^{n+1}}{2^n}\cdot\frac{|x-3|^{n+1}}{|x-3|}\cdot\frac{n^2}{(n+1)^2}$$

$$= \lim_{n\to\infty} 2|x-3|\cdot\frac{n^2}{(n+1)^2}$$

$$= |x-3|\lim_{n\to\infty}\frac{2n^2}{(n+1)^2}$$

$$= 2|x-3| < 1$$

$$|x-3| < \frac{1}{2} \leftarrow \text{Radius of Convergence}$$

$$-\frac{1}{2} < x-3 < \frac{1}{2}$$

$$\frac{5}{2} < x < \frac{7}{2}$$

Test Endpoints:

$x=\frac{5}{2}$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n 2^n(\frac{5}{2}-3)^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-2)^n(\frac{-1}{2})^n}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$

converges by p-series ($p=2>1$)

$x=\frac{7}{2}$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n 2^n(\frac{7}{2}-3)^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n 2^n(\frac{1}{2})^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$

- $\lim_{n\to\infty}\frac{1}{n^2} = 0$ ✓
- $\frac{1}{(n+1)^2} < \frac{1}{n^2}$, decreasing ✓

converges by A.S.T.

$$\boxed{\begin{array}{l}\text{R.O.C} = \frac{1}{2}\\ \text{I.O.C.}: \frac{5}{2} \le x \le \frac{7}{2}\\ \quad\text{or } \left[\frac{5}{2}, \frac{7}{2}\right]\end{array}}$$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{n! x^n}{\sqrt{n}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{n!x^n}\right| = \lim_{n\to\infty}\frac{(n+1)n!}{n!}\cdot\frac{|x|^n\cdot|x|}{|x|^n}\cdot\frac{\sqrt{n}}{\sqrt{n+1}}$$

$$= \lim_{n\to\infty}(n+1)|x|\frac{\sqrt{n}}{\sqrt{n+1}}$$

$$= |x|\lim_{n\to\infty}\frac{(n+1)\sqrt{n}}{\sqrt{n+1}}$$

$$= |x|\cdot\infty < 1 \quad \text{NEVER}$$

$$\boxed{\begin{array}{l}\text{Radius of Convergence} = 0\\ \text{Interval of Convergence}: \{0\}\end{array}}$$

5) Determine the Maclaurin series for each of the following functions. (Show all work)

(a) $f(x) = \sin(2x^2)$

$$\sin(\square) = \sum_{n=0}^{\infty} \frac{(-1)^n (\square)^{2n+1}}{(2n+1)!}$$

$$\sin(2x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x^2)^{2n+1}}{(2n+1)!}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{4n+2}}{(2n+1)!}}$$

(b) $f(x) = e^{3x} - 1$

$$e^{\square} = \sum_{n=0}^{\infty} \frac{(\square)^n}{n!}$$

$$e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n x^n}{n!} = 1 + 3x + \frac{9x^2}{2} + \dots$$

$$e^{3x} - 1 = \left(\cancel{1} + 3x + \frac{9x^2}{2} + \dots\right) - \cancel{1}$$

$$= 3x + \frac{9x^2}{2} + \dots = \boxed{\sum_{n=1}^{\infty} \frac{3^n x^n}{n!}}$$

(c) $f(x) = \dfrac{1}{2x+3}$

$$\frac{1}{1-\square} = \sum_{n=0}^{\infty} (\square)^n$$

$$\frac{1}{3\left(1+\frac{2x}{3}\right)}$$

$$= \frac{1}{3} \frac{1}{1-\left(\frac{-2x}{3}\right)} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{-2x}{3}\right)^n = \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-2)^n x^n}{3^n}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-2)^n x^n}{3^{n+1}}}$$

6) We are going to approximate $\int_0^{1/5} \cos(x^2)\,dx$ using a partial sum. Answer the following questions to find this bound; show all work.

(a) Find the Maclaurin series for the indefinite integral $\int \cos(x^2)\,dx$.

$$\cos(\square) = \sum_{n=0}^{\infty} \frac{(-1)^n (\square)^{2n}}{(2n)!}$$

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$$

$$\int \cos(x^2)\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}\,dx$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!\,(4n+1)} + C}$$

(b) Using the series from (a), find the first three terms of a series for $\int_0^{1/5} \cos(x^2)\,dx$. You do not need to simplify any expressions.

$$\int_0^{1/5} \cos(x^2)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!\,(4n+1)} \Bigg|_0^{1/5}$$

$$= \sum_{n=0}^{\infty} (-1)^n \underbrace{\frac{\left(\frac{1}{5}\right)^{4n+1}}{(2n)!\,(4n+1)}}_{a_n} \approx \boxed{\underset{n=0}{\frac{1}{5}} - \underset{n=1}{\frac{1}{2\cdot 5\cdot 5^5}} + \underset{n=2}{\frac{1}{4!\cdot 9\cdot 5^9}}}$$

(c) If we use the three terms from (b) to approximate $\int_0^{1/5} \cos(x^2)\,dx$, use the Alternating Series Remainder Estimate to bound the error between that integral and the partial sum found in (b). You do not need to simplify any expressions.

$$|R_2| \leq a_{2+1} = a_3 = \boxed{\frac{1}{6!\cdot 13\cdot 5^{13}}}$$

7) Let $f(x) = \frac{1}{x}$.

(a) Determine the Taylor polynomial of degree 2 ($T_2(x)$) for $f(x)$ that is centered at 3.

$$T_2(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2}(x-3)^2$$

$$f(x) = \frac{1}{x} \qquad f(3) = \frac{1}{3}$$

$$f'(x) = \frac{-1}{x^2} \qquad f'(3) = \frac{-1}{9}$$

$$f''(x) = \frac{2}{x^3} \qquad f''(3) = \frac{2}{27}$$

$$\boxed{T_2(x) = \frac{1}{3} + \frac{-1}{9}(x-3) + \frac{\frac{2}{27}}{2}(x-3)^2}$$

(b) If we want to bound the error when approximating $f(x)$ by the polynomial in part (a) for $2.8 \leq x \leq 3.2$ using Taylor's inequality, what is the value of M that should be used in Taylor's inequality? You do not need to simplify your answer.

$$M \geq |f^{(2+1)}(x)| \text{ on } 2.8 \leq x \leq 3.2$$

$$f'''(x) = \frac{-6}{x^4} \qquad |f'''(x)| = \frac{6}{x^4}$$

decreasing function so maximized at $x = 2.8$

$$\boxed{M = \frac{6}{(2.8)^4}}$$

8) Determine the solution of $\frac{dy}{dx} = y\cos(3x)$ where $y(0) = 2$. (Show all work)

$$\frac{dy}{y} = \cos(3x)dx$$

$$\int \frac{dy}{y} = \int \cos(3x)dx$$

$$\ln|y| = \frac{1}{3}\sin(3x) + C$$

$$y = e^{\frac{1}{3}\sin(3x) + C}$$

$$y = Ce^{\frac{1}{3}\sin(3x)}$$

$$y(0) = 2$$

$x=0$ $y=2$

$$2 = Ce^{\frac{1}{3}\sin(3\cdot 0)}$$

$$2 = Ce^{0}$$

$$2 = C$$

$$\boxed{y = 2e^{\frac{1}{3}\sin(3x)}}$$