CALCULUS II – EXAM 2 REVIEW

A review of sequences, series, and power series

SEQUENCES

- A sequence is an ordered list of numbers generated by some rule: $a_1, a_2, a_3, \ldots$
- Think of it as a function whose input is a positive integer. The main question: **does the list settle down to a single value, or not?**
- Key Concepts:
  - A sequence $\{a_n\}$ converges or narrows down to a number if $\lim_{n\to\infty} (a_n) = L$ (a finite number). Otherwise, it diverges or approaches infinity.
  - Monotonic sequences: always increasing or always decreasing.
  - Bounded sequences: trapped between an upper and lower bound.
  - Monotone Convergence Theorem: If a sequence is both **bounded** and **monotonic,** it must converge.
- There are 3 ways to express a sequence:
  - List {a1, a2, a3, …}
  - Formula: an = [], $n \geq 1$
  - Sequence Notation: $\{a_n\}^\infty$

SERIES

- A series takes a sequence and **adds it all up:** $\sum a_n = a_1 + a_2 + a_3 + \ldots + a_n$
  The big question shifts from "where does the list go?" to "does the running total settle down to a finite (non-infinite) number?

- Key Concepts:
  - The **partial sums** $S_n = a_1 + a_2 + a_3 + \ldots + a_n$ form their own sequences. If $\{S_n\}$ converges, the series converges.
  - **Geometric Series:** $\sum ar^n$ converges to $\frac{a}{1-r}$ when $|r| < 1$. Diverges if $|r| \geq 1$.
  - **P-Series:** $C \times \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, and diverges if $p \leq 1$.
  - **Alternating Series:** A series with terms that alternate. Ex. $\sum_{n=1}^{\infty} (-1)^n * \frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$
  - **Telescoping Series:** Terms cancel in pairs. Write out partial sums to find the pattern.
  - **Divergence Test (nth-term test):** If $\lim_{n \to \infty} (a_n) \neq 0$, the series diverges. If the limit IS zero, the test is **inconclusive** and you would need another test.

THE INTEGRAL TEST

- If your sequence looks like it comes from a nice continuous function, you can replace the sum with an integral. The integral and the series either **both converge or both diverge**, they are linked.
- Conditions:
  - f(x) must be **positive, continuous**, and **decreasing** on $[N, \infty)$ for some N, with $a_n = f(n)$
  - If $\int_c^\infty f(x)\,dx$ <u>converges</u> then $\sum_{n=c}^{\infty} a_n$ <u>converges.</u>
  - If $\int_c^\infty f(x)\,dx$ <u>diverge</u>s then $\sum_{n=c}^{\infty} a_n$ <u>diverges.</u>
- Be Careful:
  - You MUST check for the 3 hypothesis (in bold above).
  - Not necessary for f(x) to always be decreasing but must <u>eventually</u> be decreasing beyond some value.
  - Integral Test does NOT always work.

THE COMPARISON TESTS

- If you can't evaluate a series directly, **compare it to one you already know.**
- If your series is smaller than a known convergent series, it converges.
- If your series is bigger than a known divergent series, it diverges.
- Using a_n and b_n (A and B respectively), you can find out how to compare them ($\frac{A}{B}$)
  - If B is made to be the smallest of the two, $\frac{A}{B}$ gets <u>bigger</u>.
  - If B Is made to be the biggest of the two, $\frac{A}{B}$ gets <u>smaller.</u>
- <u>Be Careful:</u>
  - It can sometimes be difficult (or impossible) to show that the appropriate inequality holds, in which case we CANNOT use the comparison test.
  - Example:
    - $\sum_{n=1}^{\infty} \frac{4}{n^2-3n-5}$ → The series (a_n) behaves like $\sum_{n=1}^{\infty} \frac{4}{n^2}$ (b_n) which converges by P-Test.

    $\frac{4}{n^2-3n-5}$ cannot determined if it is smaller than $\frac{4}{n^2}$ so the comparison is inconclusive.

THE COMPARISON TESTS – LIMIT COMPARISON TEST

- When direct comparison is awkward, use $a_n, b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where $0 < c < \infty$, then **both series converge or both diverge**
- **Intuition:** If the ratio settles to a positive constant, the two series are "essentially the same size" for large n.
- The example from the last slide would be able to be solved with the limit comparison test.
- Example:
  - $\sum_{n=1}^{\infty} \frac{4}{n^2-3n-5}$ → The series (a_n) behaves like $\sum_{n=1}^{\infty} \frac{4}{n^2}$ (b_n) which converges by P Test.
  - $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \left(\frac{\frac{4}{n^2-3n-5}}{\frac{4}{n^2}}\right) = \lim_{n\to\infty} \left(\frac{4}{n^2-3n-5} * \frac{n^2}{4}\right) = \lim_{n\to\infty} \left(\frac{n^2}{n^2-3n-5}\right) = 1 = C$ which is between 0 and ∞
  - Therefore, the series $\sum a_n$ converges by Limit Comparison Test.

THE ALTERNATING SERIES TEST

- When terms alternate in sign (+, -, +, -,…), the partial sums zigzag toward a limit.

  This makes it easier for a series to converge since the pos and neg terms partially cancel.

- Alternating Series: $\sum(-1)^n * b_n$ or $\sum(-1)^{n+1} * b_n$ or similar with $b_n \geq 0$.

- The convergence of an alternating series will depend only on $b_n = |a_n|$.

- The Alternating Series Test or A.S.T is known as:
  - A series $\sum(-1)^n * \ b_n$with $\text{Type equation here.}$ is **eventually decreasing** (

THE RATIO TEST

- Before going into the Ratio Test, we must talk about Absolute and Conditional Convergence.
- A series $\sum a_n$ is:
  - Absolutely Convergent if: $\sum|a_n|$ converges. (This also means that regular $\sum a_n$ is convergent.)
  - Conditionally Convergent if: $\sum a_n$ converges but $\sum|a_n|$ diverges
  - Divergent if: $\sum a_n$ diverges
- Ratio Test:
  - Given the series $\sum a_n$. If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L.$
    - Then if L < 1, $\sum a_n$ is absolutely convergent.
    - If L > 1, $\sum a_n$ diverges.
    - If L = 1, then we know nothing.
  - The ratio test is inconclusive for P-Series since the $\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 1$

POWER SERIES

- A power series centered at "a" is a series with the form:
  - $\sum_{n=0}^{\infty} C_n\,(x-a)^n = C_0 + C_1(x-a)^1 + C_2(x-a)^2 + C_3(x-a)^3 + \cdots$
  - In this series, "x" is a variable and the C_n's are constants that is referred to as the coefficients of the series.
- The goal with these series is to determine the values of "x" in which the series will converge.
- There are THREE different possibilities for a power series if it is convergent (Proof w/ Ratio Test).
  1. The series converges <u>for all</u> values of x. In this case:
     a) Radius of Convergence = ∞ and Interval of Convergence = (-∞, ∞)
  2. The series converges only when x = a. In this case:
     a) Radius of Convergence = ∞ and Interval of Convergence = {a}
  3. There exists some value R > 0 (radius of convergence) such that the series converges for |x-a|< R (–R < x – a < R and a-R < x < R+a). In this case:
     a) Radius of Convergence = R and Interval of Convergence = (a – R, a + R)

POWER SERIES REPRESENTATIONS OF FUNCTIONS

- From the knowledge of geometric series, we can represent certain functions as power series.
- Geometric Series: $\sum ar^n$ converges to $\frac{a}{1-r}$ when $|r| < 1$. Diverges if $|r| \geq 1$.
- If you replace "r" with "x" and let a = 1, then we get the following:
  - $\sum_{n=0}^{\infty} 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$ provided that $|x| < 1$. This is the representation of function f(x).
- Calculus w/ Power Series
  - If a power series has radius of convergence R > 0, then the function: $f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1(x-a) + \ldots$ is differentiable (and continuous) on the interval (a-R, a+R) and:
  - The derivative is: $\frac{d}{dx}[f(x)] = \frac{d}{dx}[C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots]$
  - The integral is $\int f(x)\,dx = \int (C_0 + C_1(x-a) + C_2(x-a)^2 \ldots)dx$.

TAYLOR AND MACLAURIN SERIES

- Any "nice" function can be written as an infinite polynomial. The Taylor series matches the function's value and **all its derivatives** at a single point.
- Definitions:
  - Taylor Series of degree "N": $T_N(x) = f(a) + f'(\mathrm{a})(\mathrm{x} - \mathrm{a}) + \frac{\mathrm{f}''(\mathrm{a})(x-a)^2}{2!} + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N$
  - Maclaurin series: Taylor series centered at a = 0.
- List of Maclaurin series you must know:
  - $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\ (\text{all x})$
  - $\sin x = \mathrm{x} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\ (all\ x)$
  - $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\ (all\ x)$
  - $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots\ (|x| < 1)$
  - $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\ (-1 < x \leq 1)$
  - $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\ (|x| \leq 1)$

ADDITIONAL RESOURCES

David T. McArdle Lectures and Notes:
https://david-mcardle.scholar.uconn.edu/resources/calculus2/module2
https://david-mcardle.scholar.uconn.edu/resources/calculus2/module3

Chapter 11 Textbook Problems:
https://quizlet.com/explanations/textbook-solutions/calculus-early-transcendentals-9th-edition-9781337613927
https://quizlet.com/1149291977/calc-2-chapter-11-sequences-and-series-flash-cards/