University of Connecticut
Department of Mathematics

MATH 1132 EXAM 2 PRACTICE FALL 2024

NAME: Solutions

SIGNATURE: ____________________

Read This First!

- Please read each question carefully. The point value of each question is indicated in the box preceding its statement.
- In all free-response questions, you will *only* receive credit by providing clear work using techniques learned in this class.
- No books, formula sheets, calculators, phones, smart watches, headphones, or other references are permitted.
- No personal scrap paper is allowed.

Practice Problems Notes

- The practice problems together are almost twice as long as the actual exam.
- Timed practice without notes is essential! Try some of these under exam conditions after you have had some time to study.
- The coverage in the practice problems is not exhaustive. Look through class notes, WebAssign, and quizzes as well.

Problem 1: Indicate the validity of each statement by entering True (T) or False (F) in the blank.

(i) T The geometric series $\sum_{n=1}^{\infty} 2\left(\frac{1}{3}\right)^{n-1}$ converges to 3.

$r = \frac{1}{3} < 1$
so convergent

Converges to $\frac{\text{first term}}{1-r} = \frac{2(1/3)^{1-1}}{1-1/3} = \frac{2}{2/3} = 3$ ✓

(ii) F If the series $\sum_{n=1}^{\infty} a_n$ diverges, then the sequence a_n must diverge.

ex/ $\sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic series) diverges
but $\left\{\frac{1}{n}\right\}$ converges (to 0).

(iii) T The sequence $a_n = \frac{n^2}{e^n}$ has limit 0.

exponential functions dominate polynomials

$\lim_{n\to\infty} \frac{n^2}{e^n} = 0$

(Could use L'Hôpital's Rule twice to show this)

(iv) T The sequence $a_n = (-1)^n$ is bounded.

bounded above by 1
bounded below by -1

(v) T The partial sums of the series $\sum_{n=0}^{\infty} \frac{1}{2^n}$ are an increasing sequence.

We are adding positive terms.
$S_n = 1,\ 1+\frac{1}{2},\ 1+\frac{1}{2}+\frac{1}{4}, \ldots$

(vi) T If we use the partial sum S_5 to approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$, the Alternating Series Remainder Estimate tells us that the remainder/error satisfies $|R_5| \le \frac{1}{7}$.

$a_n = \frac{1}{n+1}$ $\quad |R_n| \le a_{n+1}$ $\quad$ so $\quad |R_5| \le \frac{1}{(5+1)+1} = \frac{1}{7}$.

(vii) F If $b_n > a_n > 0$ for all n and $\sum b_n$ diverges, then $\sum a_n$ must converge.

If larger than something finite, could be finite or infinite, inconclusive.

(viii) F The series $\sum_{n=1}^{\infty} \frac{3n-1}{5n+2}$ converges.

$\lim_{n\to\infty} \frac{3n-1}{5n+2} = \frac{3}{5} \neq 0$
diverges by Divergence Test.

Problem 2: Determine whether $\sum_{n=0}^{\infty} \frac{n^5}{n!}$ converges or diverges. Please show all work and clearly indicate any test being used.

Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^5}{(n+1)!}}{\frac{n^5}{n!}} \right| = \lim_{n\to\infty} \frac{(n+1)^5}{(n+1)!} \cdot \frac{n!}{n^5}$$

$$= \lim_{n\to\infty} \frac{(n+1)^5}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{n^5}$$

$$= \lim_{n\to\infty} \frac{(n+1)^5}{(n+1)n^5} = 0$$ (degree bigger in denominator)

So $\sum_{n=0}^{\infty} \frac{n^5}{n!}$ converges by the Ratio Test.

Problem 3: Determine whether $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+5n}}{n^3}$ converges or diverges. Please show all work and clearly indicate any test being used.

Resembles $\sum_{n=1}^{\infty} \frac{\sqrt{n^2}}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ (converges by p-series).

L.C.T.

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n^2+5n}}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sqrt{n^2+5n}}{n^{\cancel{3}\,1}} \cdot \frac{n^{\cancel{2}}}{1} = \lim_{n\to\infty} \frac{\sqrt{n^2+5n}}{n}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^2\left(1+\frac{5n}{n^2}\right)}}{n}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\sqrt{1+\frac{5n}{n^2}}}{\cancel{n}}$$

$$= \lim_{n\to\infty} \sqrt{1+\frac{5\cancel{n}}{n^{\cancel{2}}}} = \sqrt{1} = 1$$

Since $\lim_{n\to\infty} \frac{\frac{\sqrt{n^2+5n}}{n^3}}{\frac{1}{n^2}} = 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series $(p=2>1)$, $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+5n}}{n^3}$ also converges by L.C.T.

Note: Cannot use D.C.T. here (inequality doesn't work).

Problem 4: Use the Integral Test (and check all conditions) to determine whether $\sum_{n=3}^{\infty} \frac{\ln(n)}{n}$ converges or diverges. Please show all work.

Check Conditions

- positive: $n>0$ and if $n \geq 3$, $\ln(n)>0$, so $\frac{\ln(n)}{n}>0$. ✓
- decreasing: $f'(n) = \frac{\frac{1}{n} \cdot n - \ln(n)}{n^2} = \frac{1-\ln(n)}{n^2}$. If $n \geq 3$, $\ln(n)>1$ so $1-\ln(n)<0$.
  Thus $f'(n)<0$ and $\frac{\ln(n)}{n}$ is decreasing. ✓

$$\lim_{b\to\infty} \int_3^b \frac{\ln(x)}{x}\,dx = \lim_{b\to\infty} \frac{1}{2}(\ln(x))^2 \Big|_3^b = \lim_{b\to\infty}\left(\frac{1}{2}(\ln(b))^2 - \frac{1}{2}(\ln(3))^2\right)$$

∞ ; #

$= \infty$ so $\int_3^\infty \frac{\ln(x)}{x}\,dx$ diverges.

Antiderivative

$\int \frac{\ln(x)}{x}\,dx$ $u=\ln(x)$, $\frac{du}{dx}=\frac{1}{x}$, $dx = x\,du$

$= \int \frac{u}{\cancel{x}} \cdot \cancel{x}\,du = \int u\,du$

$= \frac{1}{2}u^2 + C = \frac{1}{2}(\ln(x))^2 + C$

Thus $\sum_{n=3}^{\infty} \frac{\ln(n)}{n}$ also diverges by the Integral Test.

Problem 5: What is the smallest value of N for which the Alternating Series Remainder Estimate tells us the remainder R_N for the Nth partial sum of $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ satisfies $|R_N| \leq \frac{1}{10}$? Please show all work.

$= \sum_{n=1}^{\infty} (-1)^n \underbrace{\frac{1}{\sqrt{n}}}_{a_n}$

We have $|R_n| \leq a_{n+1}$

so $|R_n| \leq \frac{1}{\sqrt{n+1}}$.

We want N such that $\frac{1}{\sqrt{N+1}} \leq \frac{1}{10}$,

so $10 \leq \sqrt{N+1}$

$100 \leq N+1$

$99 \leq N$.

$\boxed{\text{Need } N = 99}$

Problem 6: Determine whether $\sum_{n=2}^{\infty} \frac{n}{n^2-3}$ converges or diverges. Please show all work and clearly indicate any test being used.

Resembles $\sum_{n=2}^{\infty} \frac{n}{n^2} = \sum_{n=2}^{\infty} \frac{1}{n}$ diverges by p-series $(p=1 \le 1)$.

D.C.T.

$$\frac{n}{n^2-3} > \frac{n}{n^2} = \frac{1}{n}$$

smaller denom $\Rightarrow$ bigger #

Since $\frac{n}{n^2-3} > \frac{1}{n}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges by p-series $(p=1 \le 1)$, $\sum_{n=2}^{\infty} \frac{n}{n^2-3}$ also diverges by D.C.T.

Note: L.C.T. works very well for this problem.

Problem 7: Determine whether $\sum_{n=0}^{\infty} \frac{n!}{n^2+6^n}$ converges or diverges. Please show all work and clearly indicate any test being used.

Divergence Test:

$$\lim_{n\to\infty} \frac{n!}{n^2+6^n} = \infty$$

b/c Factorial functions dominate exponential and polynomial)

so $\sum_{n=0}^{\infty} \frac{n!}{n^2+6^n}$ diverges by Divergence Test.

Note: Could use Ratio Test but the "+" in the denominator makes this more challenging.

Problem 8: Determine whether the series

$$\sum_{n=2}^{\infty}(-1)^n \frac{n}{\sqrt{n^4-1}}$$

is absolutely convergent, conditionally convergent, or divergent. Please show all work and clearly indicate any test being used.

Check $\sum a_n$: A.S.T. $a_n = \frac{n}{\sqrt{n^4-1}}$

- $\lim_{n\to\infty} \frac{n}{\sqrt{n^4-1}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^4}} = \lim_{n\to\infty} \frac{n}{n^2} = 0.$
- $f'(n) = \dfrac{\sqrt{n^4-1} - \frac{n\cdot 4n^3}{\sqrt{n^4-1}}}{n^4-1} = \dfrac{\frac{\overbrace{n^4-1-4n^4}^{<0}}{\sqrt{n^4-1}}}{n^4-1} < 0.$

So a_n decreasing. Thus $\sum_{n=2}^{\infty}(-1)^n\frac{n}{\sqrt{n^4-1}}$ converges by A.S.T.

Check $\sum |a_n| = \sum_{n=2}^{\infty} \frac{n}{\sqrt{n^4-1}}$ compare to $\sum_{n=2}^{\infty}\frac{1}{n}$ diverges by p-series $(p=1\le 1)$

D.C.T.

$\frac{n}{\sqrt{n^4-1}} > \frac{n}{\sqrt{n^4}} = \frac{n}{n^2} = \frac{1}{n}.$

Since $\sum_{n=2}^{\infty}\frac{1}{n}$ diverges and $\frac{n}{\sqrt{n^4-1}} > \frac{1}{n}$, $\sum_{n=2}^{\infty}\frac{n}{\sqrt{n^4-2}}$ diverges by D.C.T.

Thus $\sum_{n=2}^{\infty}(-1)^n\frac{n}{\sqrt{n^4-1}}$ is conditionally convergent.

Problem 9: Determine whether the series

$$\sum_{n=2}^{\infty}(-1)^n \frac{n^3}{n^5+6}$$

is absolutely convergent, conditionally convergent, or divergent. Please show all work and clearly indicate any test being used.

Check $\sum |a_n| = \sum_{n=2}^{\infty}\frac{n^3}{n^5+6}$ Compare to $\sum_{n=2}^{\infty}\frac{1}{n^2}$ (Convergent p-series)

L.C.T.

$\lim_{n\to\infty}\dfrac{\frac{n^3}{n^5+6}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^5}{n^5+6} = 1.$

Since $\sum_{n=2}^{\infty}\frac{1}{n^2}$ converges by p-series $(p=2>1)$, $\sum_{n=2}^{\infty}\frac{n^3}{n^5+6}$ also converges by L.C.T.

Since $\sum_{n=2}^{\infty}\left|(-1)^n\frac{n^3}{n^5+6}\right|$ converges, $\sum_{n=2}^{\infty}(-1)^n\frac{n^3}{n^5+6}$ converges absolutely by A.C.T.

Problem 10: Determine whether $\sum_{n=1}^{\infty} \frac{n^2 7^n}{n!}$ converges or diverges. Please show all work and clearly indicate any test being used.

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^2 7^{n+1}}{(n+1)!}}{\frac{n^2 7^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2}\cdot\frac{7\cdot\cancel{7^n}}{\cancel{7^n}}\cdot\frac{\cancel{n!}}{(n+1)\cancel{n!}}\right|$$

$$= \lim_{n\to\infty}\frac{7(n+1)^2}{(n+1)n^2} = 0$$ (degree bigger on bottom)

So $\sum_{n=1}^{\infty}\frac{n^2 7^n}{n!}$ converges by the Ratio Test.