Chapter 11 – Sequence, Series, and Power Series

Hey y'all, it's Akeva! This document is going to be focusing on sequences, series, and power series. I will be honest with you all, this unit had to be my least favorite and the hardest for me to grasp conceptually. However, no worries, as someone who is stubborn and goes through resources until I understand things, I will take my time to give y'all videos that helped me and go through my thought process in approaching these problems. If you study with music, put on some music that will hype you up and let's get into this.

Since this my first resource for Calculus II, I want to take some time going into how I studied for this class and some things that will set you up for success.

1. I took this class online (😬- only do if you have the discipline for it) but with David McArdle. If you have a chance to take it with him or sit in on his classes, I highly suggest you do so. As someone who thought Calc II was the hardest math class among the Calc classes and Linear Algebra, he made my life so much easier. He used to have his videos easily accessible on his website. It's now password protected. So, I would suggest emailing him for the password and explain you would like extra resources or simply ask someone who is in his class.
2. I won't sit here and tell you not to procrastinate because I may be the Command-in-Chief of doing so. However, I would say give yourself at least 1 ½ days before a quiz and at least 2-3 days before an exam where you review content for at least 2+ hours a day. This allows you to go over the necessary content and do many practice problems, which is key.
    a. Redo quiz problems and questions done in lectures.
    b. Do extra problems on YouTube or anywhere that gives you the answers with detailing step-by-step how they go their answer. To be honest, I would not fully trust ChatGPT as sometimes it messes up series and its logic does not necessarily always make sense to someone who is totally lost. There are also ample resources online where this is not necessary.

This is all I can think about for right now. If anything, else pops up, I'll be sure to add it!

Alright, let's get into this. If you are not the type of person who likes reading to learn (me), feel free to pop this PDF into NaturalReader and listen to the document while pausing to look at figures or even watch videos.

Sequences

There are sequences that come up again and again that are imperative for you to know:

$$a_n = \frac{1}{2^n}$$

$$a_n = \frac{n}{n+1}$$

Types of Problems with Sequences

1. Determining sequence equation based on the first n terms
2. Finding the limit of a sequence
3. Determining monotonic and bounded sequences

How to determine the sequence based on the first n terms

I will give some helpful tips that should shape your understanding in these problems:

- If you see alternating negative and positive terms, you have a term with $(-1)^n$
  - To determine if you should start with n=0 or n=1, see if the first number in the sequence is negative or positive. If it is negative, you either start with n=1 for the full sequence (if this does not affect the other terms) or, if it does, make the exponent be n + 1. I will do out a problem to better explain the distinction
- Determine if the numerator is the same across all values or if it is increasing
  - If it is increasing, by what amount? Does it hold true for all numbers in the sequence? If so, then you have your numerator
  - If it is the same across all values, there is your numerator
- Look at your denominator, is it increasing rapidly or at a constant rate
  - If it is increasing rapidly, you have an exponential somewhere in your denominator
    - To determine the base, you will have be familiar with exponents and quickly determine them. Good news is that I would not anticipate any exponents higher than base 5. A good hint at what the base may be is the first denominator in the sequence. If it is 1, then look at the second number in the sequence.
  - If it is increasing at a constant rate, then similarly to numerator, think about by what amount it increases and what serves as the basis to that increase. As n increases, is there an extra term?
- Now multiply your result together

Let's do an example problem together with this logic:

Given the sequence $\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \ldots\}$, find a formula.

Alright, so I see alternating negatives and positives, meaning I have some form of $(-1)^n$. Since it starts positive, I am inclined to believe that it may take the form n, and n starts at 0 (since even a negative number to the power of 0 is positive 1).

Now, let's go to the numerator. We start with a 3 and seem to increase with a value of 1. Since I am tentatively starting my n at 0, it might be n + 3. Okay, let's keep that in place for now.

Going to the denominator, I see that the first denominator is 5 and then 25. I immediately think of base 5 since 5 to the power of 2 is 25. And that trend holds for the third denominator which equals 5 to the 3rd power. But since the first denominator is 5, I think that we need to start at n =1 and not 0.

Okay, backtracking now, the numerator should be n + 2 now and the negative term should be n + 1 or n -1 (does not matter because we would get the same trend in the exponents).

Let's put everything together now. My sequence will be $(-1)^{n+1}\frac{n+2}{5^n}$, with n starting at 1 and going to infinity.

Hopefully that made sense, if not and/or for more practice, use the following links:

- Organic Chemistry Tutor
- Krista King
- Michel van Biezen

How to determine the limit of a sequence

Before we begin, let's talk about some definitions as a refresher or something new (if it's your first time seeing this):

Convergent: as n increases to infinity in a sequence, the sequence approaches a final number, L

Divergent: as n increases to infinity in a sequence, the sequence does not approach a finite value

Now, let's address the question, what are some of the properties of convergent sequences? Some of them may be intuitive but may sure you know them as you may be tested explicitly or implicitly on them. I will write things first in layman's terms and then the mathematical terms out:

1. If we are taking the limit of a function f(x) where x approaches infinity and it equals L, and we know that f(n) = sequence we are interested in, then the limit of the sequence as n approaches infinity is L as well. You can think of it as changing the variables. Since the limit of f(x) approaching infinity is convergent, so will f(n) because it's just the same function but with different variables.
    a. If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, where n is an integer, then $\lim_{n\to\infty} a_n = L$.
2. If the limit of sum of two sequences defined in n, and n is taken to infinity, the limit is equivalent to the sum of the separate limits for the sequences. It's exactly like taking a derivative or integral of $x + x^2$ and we can take the separate derivatives or integrals and sum the results together as we did in Calculus I

a. $\lim_{n\to\infty} f(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

3. It's the same thing as point number 2 but with a minus sign. The limit of the difference of two sequences defined in n is the same as the difference of separate limits.
    a. $\lim_{n\to\infty} f(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
4. If there is a constant attached to a sequence as we are trying to take the limit, we can take the constant out, such that c is multiplied to the result of the limit of the sequence as n goes to infinity
    a. $\lim_{n\to\infty} f(ca_n) = \mathrm{c} \lim_{n\to\infty} a_n$
5. The limit of the product of two sequences is equal to the product of the separate limits for each sequence.
    a. $\lim_{n\to\infty} f(a_n b_n) = \lim_{n\to\infty} a_n * \lim_{n\to\infty} b_n$
6. The limit of the quotient of two sequences is equal to the quotient of the separate limits for each sequence.
    a. $\lim_{n\to\infty} f\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$
7. The limit of a sequence taken to a power, p is equal to the limit of the sequence raised to the p power. This is only true when p is greater than 0 and the sequence is greater than 0.
    a. $\lim_{n\to\infty} {a_n}^p = \left[\lim_{n\to\infty} a_n\right]^p \; if\ p > 0\ and\ a_n > 0$
8. The next one is based on the Squeeze Theorem which essentially allows us to find a function's limit by trapping it between two other functions whose limits are defined. Therefore, if sequence a is less than or equal sequence b which is less than or equal to sequence c, for an n greater than n_o (a natural number index acting as a threshold where the inequalities hold) and the limit of sequence a as n approaches infinity is equal to the limit of sequence c approaching infinity, which is also equal to L, then, sequence b must equal L. This is because b is sandwiched between a and c and are equal to L already. To make the former inequality hold, b must be L.
    a. $If\ a_n \leq b_n \leq c_n\ for\ n \geq n_o\ and \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L, then \lim_{n\to\infty} b_n = L$
9. If the limit of the absolute value of a sequence as it approaches infinity is equal to 0, then the limit of the regular sequence also equals 0. This is because the positive and negative terms will ultimately cancel each other out, making the most important part of the sequence, the part without the alternative positive and negative terms, crucial for convergence.
    a. If $\lim_{n\to\infty} |a_n| = 0, then \lim_{n\to\infty} a_n = 0$
10. If the limit of a sequence as it approaches infinity equals L and a function f, is continuous at L, then the limit of the function of the sequence as it approaches infinity equals the function of L. You can treat it as whatever you do to one side of the equation, you have to do to the other. This means that since we are taking the function of the sequence, we do the same to the other side (which is wide the function has to be continuous at L).

a. If $\lim_{n\to\infty} a_n = L \; and \; the \; function \; f \; is \; continous \; at \; L, then \lim_{n\to\infty} f(a_n) = f(L)$

11. The limit of a sequence defined as a number to the nth power approaches infinity equals 0 when the number (doesn't need to be an integer) is greater than -1 and less than 1. The limit equals 1 when r=1. This means r can be a half, for example, and it would converge to 0.
    a. $\lim_{n\to\infty} r^n = \begin{cases} 0 & if -1 < r < 1 \\ 1 & if \; r = 1 \end{cases}$

For solving if a sequence is convergent, you will have to draw back to your knowledge of L'Hôpital's rule that you should have covered in Calc I. For a quick refresher, refer to this video or this one (longer review with more trig identities) by the Organic Chemistry Tutor.

The way I like to approach these problems is the following:

1. What are the powers of the numerator and denominator for the variable approaching infinity?
    a. If they are both equal, use L'Hôpital's Rule to find it is convergent or divergent
    b. If the denominator is larger, the sequence will converge to 0
    c. If the numerator is larger, the sequence will diverge aka go off to infinity
2. Depending on you professor, they may take the logic of if the numerator is larger or denominator is larger, therefore… However, if they do not, default to using L'Hôpital's rule.

To help with recognizing if something is larger than the other, here are some rules:

- If you encounter an exponential and power function, know that an exponential will beat out a power function (ex: x squared)
- A power function beats out a constant
- Higher power beat out lower powers
- With two exponentials, such as e to the x or 2 to the x, go with the base that is larger. So, for this example, e ~ 2.718 so it will beat out base 2. However, it would not win against 3^x.

For practice on finding limits of sequences, refer to the following videos:

- Organic Chemistry Tutor
- Professor Leonard
- Kimberly Brehm

Monotonic and Bounded Sequence

Let's move on to Monotonic and Bounded Sequences! New section means new definitions:

Increasing: a sequence is defined as increasing if $a_n < a_{n+1} \; for \; all \; n \geq 1$

Decreasing: when $a_n > a_{n+1}\ for\ all\ n \geq 1$ when $a_n\ is\ a\ sequence$

Monotonic: when a sequence is either increasing or decreasing. You can remember this because mono means one so one tone aka either increasing or decreasing.

Bounded above: a sequence is bounded above is there is a number M, such that the sequence is less than or equal to M for all values greater than 1

Bounded below: if there is a number m, such that the sequence is greater than or equal to m for all values greater than 1

Bounded sequence: a sequence that is bounded above and below

Note: if a sequence is bounded above and increasing, bounded below and decreasing, or just bounded, it's convergent.

Generally, the types of problems you will see are:

- Determining if a sequence is increasing or decreasing
    - In this case, list out the first couple of terms. Note what is growing and/or the quickest. If it's the denominator, it's decreasing. If it's the numerator, the function is increasing
    - In terms of showing this. Rewrite the second term in terms of the first
        - Example:

$$a_n = \frac{3}{n+5}$$

$$a_n = \frac{3}{n+5} > \frac{3}{n+6} = \frac{3}{(n+5)+1} = a_{n+1}$$

- Determining if a sequence is bounded, bounded above, or bounded below
    - This loops back to converging and diverging. If it converges, you will need to determine if it is increasing or decreasing to see if it's bounded above or below. For being bounded, try the first couple of terms, if the terms are very similar, they are most likely bounded

For practice and more knowledge on the topic, refer to the following videos:

- Organic Chemistry Tutor
- Kimberly Brehm
- Math with Professor V

Series

Series Definition

Before we dive into series and the types of problems you will encounter, let's start with some definitions and examples of series to ground ourselves:

Series aka infinite series: the sum of all the terms in a sequence. This is denoted using $\sum_{n=1}^{\infty} a_n$.

- If the series is convergent, then $\lim_{n\to\infty} a_n = 0$

Partial sums: a smaller sequence of the larger sequence from 1 to n terms. This is mainly used to see if the general series has a sum.

- $s_1 = a_1$
- $s_2 = a_1 + a_2$
- $Generally, s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$

Some examples of series that are crucial and will show up again and again in Calc II are the following:

- Geometric series: a series where each term is obtained from the preceding one by multiplying it by a common ratio r
  - $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$
  - It is convergent if the absolute value of r is less than 1 and its sum is $\frac{a}{1-r}$. Else it is divergent
  - Note that a is the first term of the series. To find r, divide the third term, ar^2 by the second term, ar $(\frac{ar^2}{ar} = r)$.
- Harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n}$
  - This series is divergent at the denominator does not increase quick enough to make the series converge to a number.
  - This would increase any variant which would mean any series whose numerator stays the same, yet the denominator may be offset by a value such as n + 3

Properties of Convergent Series

It's imperative to know the properties as they can greatly help you solving problems. Below are key properties:

- Similar to sequence, if there is a constant in the series, you can multiple the constant to the result of the infinite series
  - $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$
- If you are taking the sum of two sequences a and b, from n=1 to infinity, it is equal to the sum of series of each sequence
  - $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

- If you are taking the difference of two sequences a and b, from n=1 to infinity, it is equal to the difference of series of each sequence
  - $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Types of Problems with Series

Some problems you will get dealing with series are the following:

- Finding the sum of geometric series
  - To solve these problems, you need to determine a and r for the geometric series
  - Practice videos:
    - Organic Chemistry Tutor (infinite)
    - Organic Chemistry Tutor (finite aka partial sums)
    - TEACHER MJ – MATH TUTORIAL
- Determining if a series is convergent or divergent, which can involve transforming a series into a geometric series
  - To solve these problems, transform the series into an equation that you can recognize as a geometric series. If r ends up be greater than 1, we know that is diverges. There are also other ways to determine if a series converges or diverges, which will be covered in a later section of this PDF
  - Practice video:
    - Recognizing series vs sequences and arithmetic series vs geometric series: Organic Chemistry Tutor
- Word problems that utilize a geometric series
  - Practice video:
    - Oninab Resources
    - Ryan Meyering
- Writing a number as a ratio of integers (hint hint geometric series)
  - Practice videos:
    - Krista King
    - Khan Academy
- Testing for divergence
  - This is based on the limit of the sequence as n grows to infinity. If it does not exist or the limit does not equal 0, then the series is divergent. This does not mean that the series converges to 0. However, that's where we need further testing
  - If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n$ 0, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
  - Practice video:
    - Organic Chemistry Tutor
    - Khan Academy

Types of Convergence Tests

Remember how I mentioned that the divergence test only tests that and doesn't give you the exact sum of an arithmetic series (not a geometric series). So, there are several tests you can conduct. I'll go through all of them and then give a resource that can help you decided which one to use if not specified on an exam.

- Integral Test
  - It's exactly as you think. You take an integral of $f(n) = a_n$ from 1 to infinity. If it is convergent, then the series is convergent. If it is divergent, then the series is divergent. It does not necessarily need to start at 1. It starts wherever the series is defined as.
  - Practice videos:
    - Organic Chemistry Tutor
    - Professor Dave Explains
    - Professor Leonard (longer video)
  - To find an approximation to the sum of the series, we need to find the size of the remainder:
    - Defined as $R_n = a_{n+1} + a_{n+2} + \cdots \leq \int_n^\infty f(x)dx$
    - The Remainder Estimate for the Integral Test
      - Supposing $f(k) = a_k$, where f is a continuous, positive, and decreasing function for $x \geq n \; and \; \sum a_n$ is convergent. If $R_n = s - s_n$, then $\int_{n+1}^\infty f(x)\,dx \leq R_n \leq \int_n^\infty f(x)\,dx$
  - Practice problems for this:
    - Organic Chemistry Tutor
    - James Hamblin
- Comparison Tests
  - The general idea behind the comparison test is that we compare a given series with another series that is known to be convergent or divergent. If we compare the terms to see which is larger (Direction Comparison Test) or the limit of the ratios (Limit Comparison Test)
  - Direct Comparison Test
    - Let's say we are given two series, a and b (n omitted for conciseness) and they have positive terms. If b is convergent and sequence of a is less than or equal to the sequence of b for all n, then the series a is also convergent. If b is divergent and the sequence a is greater than or equal to the sequence of b, for all n, then the series a is also divergent
    - Practice problems:
      - Organic Chemistry Tutor
      - JK Math

        - To estimate the sums, we do something very similar to the integral test with a remainder
            - Defined as $R_n = a_{n+1} + a_{n+2}$ and $T_n = b_{n+1} + b_{n+2}$ for the comparison series
- Limit Comparison Test
    - Let's say that we are once again given two series a and b with positive terms. If the limit of a over b where n approaches infinity equals c, where c is a finite number and greater than 0, then both series are convergent of divergent.
    - Practice problems:
        - Organic Chemistry Tutor
        - JK Math
- Alternative Series Test
    - If you have terms that alternative between positive and negative terms, this the test you should using as the ones discussed so far only apply to positive terms
    - The test states that if the series satisfies the following conditions, it is convergent:
        - $b_{n+1} \leq b_n \; for \; all \; n$
        - $\lim_{n \to \infty} b_n = 0$
    - To estimate the series, we use the Alternating Series Estimation Theorem which states if $s = \sum(-1)^{n-1} b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies the following:
        - $b_{n+1} \leq b_n$
        - $\lim_{n \to \infty} b_n = 0$
        - then, $|R_n| = |s - s_o| \leq b_{n+1}$
    - Practice problems:
        - Organic Chemistry Tutor
        - Professor Leonard
        - JK Math
    - Absolutely convergent: a series is absolutely convergent if the absolute value of the series is convergent
        - If it is absolutely convergent, then it is convergent
        - If a series is absolutely convergent with sum s, then any rearrangement of the series has the same sum
            - Rearrangement: a series obtained by changing the order of the terms
    - Conditionally convergent: if the series is convergent but the absolute value of the series diverges

        - Practice videos for absolute and conditional convergence:
            - Organic Chemistry Tutor
            - JK Math
            - Professor Leonard
    - Ratio Test
        - Useful for determining if a series is absolutely convergent
        - The ratio tests basically says that if the ratios $\left|\frac{a_{n+1}}{a_n}\right|$ approach a number less than 1 as n approaches infinity, then the series converges
        - The formal definition consists of the following conditions:
            - If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent)
            - If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent
            - If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, you can't use the Ratio Test to see the series are convergent or divergent
        - Practice videos:
            - Organic Chemistry Tutor
            - Professor Leonard (almost 2 hours)
            - JK Math
    - Root Test
        - You use this test when the nth powers occur
        - The official definition consists of the following conditions:
            - If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$, the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent)
            - If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$ or If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent
            - If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$, you can't use the Ratio Test to see the series are convergent or divergent
        - Practice videos:
            - Organic Chemistry Tutor
            - JK Math

Okay, we've gone through a bunch of tests. So how do we know which one to use when given a series?

For my readers, refer to Section 11.7 in your textbook or the guide posted to the website made by one of the professors here at UConn. For those who like learning through videos, refer to the following videos for a review and to know which tests to use:

- Organic Chemistry Tutor
- Mu Prime Math
- Dr. Trefor Bazett (very peppy)

Power Series

Okay, we have covered a lot so far so nice job to you! We are going to start the last major section for Chapter 11, power series. It may take a minute for you to grasp the content so be sure to give yourself time and grace as this was one of the hardest sections for me as well. Take a deep breath and let's jump into it. I'll try my best to explain topics and as usual provide resources!

Naturally, with a new topic, we have to define things. What is a power series?

Power series: series in the form $\sum_{a=0}^{\infty} c_n x^n = c_o + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$ where x is a variable and c_n are constants

- For each number we substitute for x, the series turns into a series of constants that we can test for convergence of divergence. This is similar the series we were discussing above. For the sake of the example, image that x is equal to 1 and the constants follow a sequence of numbers
- The sum of series is a function where $f(x) = c_o + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$, whose domain (span of x values) is the set of all x for which the series converges
- The power series can also take another form where it is centered at a: $\sum_{n=0}^{\infty} c_n (x-a)^n = c_o + c_1(x-a) + c_2(x-a)^2 + \cdots$

Note that for power series, to determine convergence, we normally use the Ratio or Root Tests.

As I mentioned before, a power series only converges for a set of x values and it's imperative that we know that domain. There are only three possibilities:

- The series converges only when x = a
- The series converges for all x
- There is a positive number R, such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$
  - The number R referred here is called the radius of convergence of the power series. Let's see if this makes sense to you. What should be the R for the first and second cases?
  - ….For the first case, it should be R=0 since there is only one number for which the series converges. For the second $R = \infty$.

As a branch off the radius of convergence, you also have the interval of convergence which is the interval that consists of all values of x for which the series converges. So, what would be the radius of convergence for the first and second cases?

…Just a single point, a, and $(-\infty, \infty)$, respectively. With regards for the third condition, we have to do some equation manipulation to find it. Can you figure it out? If not, I'll do it below:

$$|x - a| < R$$

$$-R < x - a < R$$

$$-R + a < x < R + a$$

As a little reminder when it comes to absolute values and inequalities, we are technically removing or not regarding negative terms. This means when looking at the full inequality and getting rid of the absolute values, we must at the negative back. We can also add or subtract integers, but we must do it to all sides of an inequality or else we are changing the inequality.

With the third condition, when x is the endpoint of the interval aka $x = a \pm R$, anything can happen, meaning that the series could:

- Converge at one of the endpoints:
  - $(a - R, a + R]$
  - $[a - R, a + R)$
- Converge at both endpoints:
  - $[a - R, a + R]$
- Or diverge at both endpoints
  - $(a - R, a + R)$

Where parentheses mean we are not including that value and square brackets mean we do.

Remember when I mentioned that we use the Ratio and Root tests? We use it to determine the radius of convergence R. However, they ALWAYS fail when x is an endpoint of the interval of convergence. Therefore, we need to check the endpoints with another test, like the ones mentioned above in the past section of this document.

Here are some practice problems on the topic:

- Organic Chemistry Tutor
- JK Math

Geometric Series

Based on our newfound knowledge, we must figure out how to represent functions as series which is useful integrating functions without elementary antiderivatives and approximating functions as polynomials. Let's start with the geometric series and see how we use it to find the power series of other functions:

Geometric series: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \ where \ |x| < 1$

Based on this, we can determine the power series for $1/(1+x^2)$ and the interval of convergence

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

Essentially, we manipulate the equation to take the form of the geometric power series and simplify. We see that there is a negative in front of x squared so there is an alternating term. Where x was, there is a negative x squared which was extracted. For the interval of convergence, we know for normal geometric power series, the interval of convergence was $(-1,1)$ based on it being valid for values $|x| < 1$. So, let's add manipulate this fact for the purposes of this problem:

$$|-x^2| < 1$$

$$x^2 < 1$$

$$|x| < 1 = -1 < x < 1$$

Which is the same as (-1, 1)!

For more practice on this, refer to the following videos:

- Organic Chemistry Tutor
- Math with Professor V
- blackpenredpen
- JK Math

When we want to integrate and differentiate these power series, we need to keep in mind the following:

- If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence R > 0, then the function f is differentiable and continuous on the interval (a – R, a + R) and
  - $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$
  - $\int f(x)\,dx = C + c_o(x-a) + \frac{c_1(x-a)^2}{2} + \frac{c_2(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} \frac{(x-a)^{n+1}}{n+1}$
- The radii of convergence of the power series in the above equations are both R aka they do not change

For practice with differentiating and integrating, refer to the following videos:

- Organic Chemistry Tutor
- Math with Professor V
- JK Math

Taylor and Maclaurin Series

Other than the geometric series, there are two other power series that are widely used and you will encounter a lot. They are the Taylor and Maclaurin Series. Let's define them:

- Taylor series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

  - You can like of the $c_n s$ being defined as $c_n = \frac{f^{(n)}(a)}{n!}$ where $f^{(n)}$ are the nth derivatives of

$$f(x) = c_o + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots for\ |x-a| < R$$

- Maclaurin Series (a special case of the Taylor series where x=0)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

For practice with finding the Taylor series based on a given function f(x), look at the following videos:

- vinteachesmath
- Organic Chemistry Tutor

As a next step to these types of problems, we also need to determine whether a function has a power series representation aka under what circumstances is a function equal to the sum of its Taylor series? To do this, we need to define two key things:

- Nth-degree Taylor polynomial of f at a

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^n(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- Remainder of the Taylor series

$$R_n(x) = f(x) - T_n(x)$$

$$f(x)\ is\ defined\ as\ \lim_{n\to\infty} T_n(x)$$

With these terms, we have the following theorem:

- If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a, and if $\lim_{n\to\infty} R_n(x) = 0$ for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $|x - a| < R$

To show that $\lim_{n\to\infty} R_n(x) = 0$, we use Taylor's Inequality Theorem:

- If $\left|f^{(n+1)}(x)\right| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1} \quad for\ |x - a| \leq d$$

As helpful tip for solving these problems, be sure to know the following:

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad for\ every\ real\ number\ x$$

For practice with Taylor's inequality, use the following resources:

- Organic Chemistry Tutor
- Math with Professor V
- Krista King

There are some Maclaurin Series you should know/be able to recognize. I believe they should give you key ones. However, on the offhand they do not, some are below:

Five Basic Maclaurin Expansions

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 \cdots = \sum_{n=0}^{\infty} x^n$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

Calcworkshop.com

Binomial Series

There is also another series that are imperative that you should know: the Binomial Series. Its form is the following:

- If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

Where $\binom{k}{n} = \frac{k(k-1)(k-2)\ldots(k-n+1)}{n!}$

The binomial series always converges when $|x| < 1$, we will need to determine if it converges on the endpoints. Luckily, this is defined for us:

- The series converges at 1 if $-1 < k \leq 0$
- The series converges at both endpoints' if $k \geq 0$

Part of the problems you should expect to see in addition to the ones listed in this sheet are obtaining new Taylor Series from Old Series. They follow a similar strategy I did very early on in this section. Here is some more practice:

- Dr. Trefor Bazett
- Sarah Schott
- Professor Leonard
  - Use the following time points for what we need (taken from a comment):
    - 0:00 Taylor & Maclaurin Series
    - 30:00 Example
    - 45:30 Example
    - 1:14:30 Example
    - 1:39:49 Example
    - 1:51:00 Binomial Series
    - 2:00:00 Example
    - 2:12:00 Common Maclaurin Series
    - 2:21:30 Example
    - 2:38:10 Example
    - 2:54:50 Example

As a note, we can multiply and divide power series, which can be ways you get new Taylor series from old ones! You would focus on the on the first terms as they are most important. If you need a refresher in doing long division with polynomials, refer to the following video:

- Organic Chemistry Tutor

For extra videos (some of the above videos have some examples of this):

- Chris Chappa
- Lorenzo Sadun

That's the end of the general content of Chapter 11, which should cover most, if not all, of the topics that are covered and will appear on Exam 2! If you are responsible for 11.11 (Applications of Taylor Polynomials), I would refer to this video by Math with Professor V! Good luck studying and feel free to reach out or come in during Math help hours if need be!