MATH 1132Q
Module I Review

A review of Integration Techniques

Integration by Parts

$\int u\,dv = \mathrm{uv} - \int v\,du$

- When to use IBP
  - Polynomial times something else (ex. xe^x, xlnx, xsinx)
  - Logarithm or inverse trig function (ex. lnx, arctanx, arcsinx)
  - A product where:
    - one part (u) gets simpler when differentiated (ex. 2x^2)
    - the other part (v) is easy to integrate (ex. e^x)
- How to choose u (LIATE)
  - 1. Log → lnx
  - 2. Inverse Trig → arctanx
  - 3. Algebraic → x, x^2
  - 4. Trig → sinx, cosx
  - 5. Exponential → e^x

Integration by Parts – Table Method

- Need to integrate more than once? Use this (HUGE time-saver):

Sign	Differentiate		Integrate
+	u	+	dv
-	u’	-	v
+	u’’	+	$\int v$
-	u’’’	-	$\int\int v$
+	…		…

STOP once u column = **0**

- Green arrow → **Multiply** terms together by +1 or -1

Integration by Parts - Example

$\int u\,dv = uv - \int v\,du$

- 1. Integrate ∫ x² lnx dx

Using LIATE:

- $u = \ln x$ (logarithmic wins)
- $dv = x^2 dx$

$$du = \frac{1}{x}dx, \quad v = \frac{x^3}{3}$$

$$\int x^2 \ln x\, dx = \frac{x^3}{3}\ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3} \cdot \frac{x^3}{3}$$

$$\boxed{\int x^2 \ln x\, dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C}$$

Integration by Parts – Longer Example

$\int u\,dv = uv - \int v\,du$

- 2. Integrate ∫ x^3e^xdx

$$u = x^3, \quad dv = e^x dx$$

$$du = 3x^2 dx, \quad v = e^x$$

$$\int x^3 e^x dx = x^3 e^x - \int 3x^2 e^x dx$$

$$= x^3 e^x - 3 \underbrace{\int x^2 e^x dx}$$

$$= x^3 e^x - 3 \left(x^2 e^x - \underbrace{\int 2x e^x dx} \right)$$

$$= x^3 e^x - 3 \left(x^2 e^x - \left(2x e^x - \int 2e^x dx \right) \right)$$

$$= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C$$

$$\boxed{= e^x (x^3 - 3x^2 + 6x - 6) + C}$$

Integration by Parts – Longer Example w/ Table Method

- We get the same answer but WAY quicker

- 2. Integrate ∫ x^3e^xdx

Sign	Differentiate	Integrate
+	x^3 +	e^x
-	3x^2 -	e^x
+	6x +	e^x
-	6 -	e^x
+	0 (stop)	e^x

$$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x$$

Integration using Trig Identities

- Useful trig identities

$$\sin^2 x = 1 - \cos^2 x$$
$$\cos^2 x = 1 - \sin^2 x$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$\sin(2x) = 2 \sin x \cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x$$

- 4 cases
  - Odd powers (ex. sin^3(x))
  - Even powers (ex. cos^2(x))
  - Mixed trig powers (ex. sin^2(x)cos^2(x))
  - Products (ex. sinxcosx)

Odd Powers Case

$$\int \sin^3 x \, dx$$

$$\sin x \cdot \underline{\sin^2 x}$$

$$\sin^2 x = 1 - \cos^2 x$$

$$\sin^3 x = \sin x(1 - \cos^2 x)$$

Now substitute:

- $u = \cos x$

Even Powers Case

$$\int \cos^2 x \, dx$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$\int \cos^2 x dx = \int \frac{1 + \cos(2x)}{2} dx$$

$$\boxed{= \frac{x}{2} + \frac{\sin(2x)}{4} + C}$$

Mixed Trig Case

$$\int \sin^2 x \cos^2 x \, dx$$

1st Identity

$$\sin^2 x = \frac{1-\cos(2x)}{2}, \quad \cos^2 x = \frac{1+\cos(2x)}{2}$$

$$= \frac{(1-\cos(2x))(1+\cos(2x))}{4} \quad = \frac{1-\cos^2(2x)}{4}$$

2nd Identity

$$\cos^2(2x) = \frac{1+\cos(4x)}{2}$$

Substitute back in

$$\sin^2 x \cos^2 x = \frac{1}{4}\left(1 - \frac{1+\cos(4x)}{2}\right)$$

$$= \frac{1}{4}\left(\frac{2-1-\cos(4x)}{2}\right)$$

$$\boxed{= \frac{1-\cos(4x)}{8}}$$

Trig Products Case

$$\int \sin x \cos x \, dx$$

$$\sin x \cos x = \frac{1}{2} \sin(2x)$$

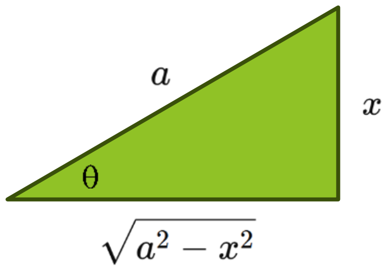
$$\int \sin x \cos x dx = \frac{1}{2} \int \sin(2x) dx$$

Now u-sub can be used with u = 2x

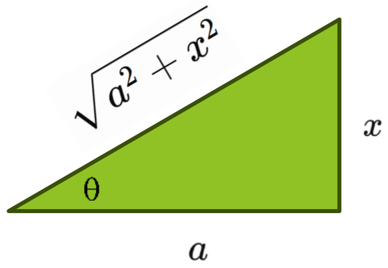
Integration by Trig Substitution

- Used for integrals that contain these three cases

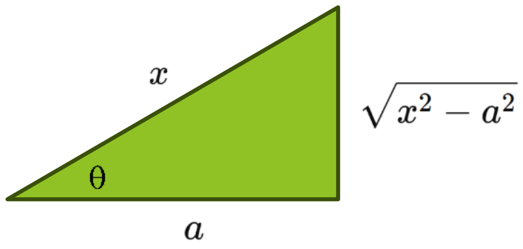
$$\sqrt{a^2 - x^2} \Rightarrow x = a\sin\theta$$



$$\sqrt{a^2 + x^2} \Rightarrow x = a\tan\theta$$



$$\sqrt{x^2 - a^2} \Rightarrow x = a\sec\theta$$



tan case → everything turns into sec

sec case → everything turns into tan

sin case → everything turns into cos

Integration by Trig Substitution – tan case

- Integrate $\displaystyle\int \frac{1}{(x^2+4)^{3/2}}\,dx$

STEP 1

Find x

$$x^2+4 = a^2+x^2 \Rightarrow x = 2\tan\theta$$

Plug in x

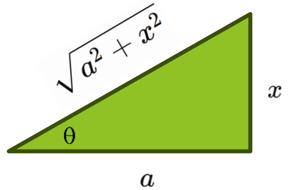
$$dx = 2\sec^2\theta\,d\theta$$

Save for later

STEP 2

Draw triangle

$$\sqrt{a^2+x^2} \Rightarrow x = a\tan\theta$$



STEP 3

Sub in x & dx

$$x^2+4 = 4\tan^2\theta+4 = 4\sec^2\theta$$

$$(x^2+4)^{3/2} = (4\sec^2\theta)^{3/2} = 8\sec^3\theta$$

Original: $\displaystyle\int \frac{1}{(x^2+4)^{3/2}}dx$

Now: $\displaystyle\int \frac{1}{8\sec^3\theta}\cdot(2\sec^2\theta\,d\theta) = \int\left(\frac{2}{8}\right)\boxed{\left(\frac{\sec^2\theta}{\sec^3\theta}\right)}d\theta$

$$= \frac{1}{\sec\theta} = \cos\theta$$

STEP 4

Use triangle

Simplify + Integrate: $\displaystyle\int \frac{1}{4}\cos\theta\,d\theta = \frac{1}{4}\sin\theta + C$

From triangle (**SOH**-CAH-TOA):

$$\sin\theta = \frac{x}{\sqrt{x^2+4}}$$

$$\boxed{\frac{x}{4\sqrt{x^2+4}} + C}$$

Integration by Trig Substitution – sin case

- Integrate $\int_0^4 \frac{x^2}{\sqrt{16-x^2}}\,dx$

$16 - x^2 \Rightarrow x = 4\sin\theta \longrightarrow \underline{x^2 = 16\sin^2\theta}$ = $16(1-\sin^2\theta) = 16\cos^2\theta$

$dx = 4\cos\theta\,d\theta$

So, $\underline{\sqrt{16-x^2} = 4\cos\theta}$

In terms of theta now,
so **change bounds**!

$x = 0 \Rightarrow \theta = 0$

$x = 4 \Rightarrow \theta = \frac{\pi}{2}$

Plug in

$$\int_0^{\pi/2} \frac{16\sin^2\theta}{4\cos\theta}\cdot 4\cos\theta\,d\theta$$

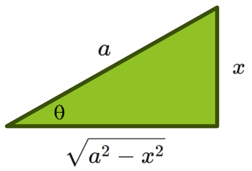
$$= \int_0^{\pi/2} 16\underline{\sin^2\theta}\,d\theta \quad = 16\cdot\frac{1}{2}\int_0^{\pi/2}(1-\cos(2\theta))d\theta$$

Identity: $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$

$$= 8\left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{\pi/2}$$

$$\boxed{= 8\cdot\frac{\pi}{2} = 4\pi}$$

$\sqrt{a^2-x^2} \Rightarrow x = a\sin\theta$



Integration by Trig Substitution – sec case

- Integrate $\displaystyle\int \frac{1}{x^2\sqrt{x^2-64}}\,dx$

$$x^2 - 64 \Rightarrow x = 8\sec\theta \longrightarrow \sqrt{x^2-64} = 8\tan\theta \longrightarrow \int \frac{8\sec\theta\tan\theta}{64\sec^2\theta\cdot 8\tan\theta}d\theta$$

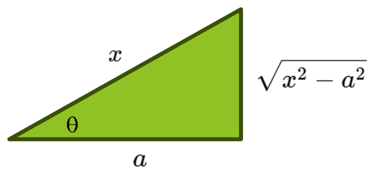
$$dx = 8\sec\theta\tan\theta\,d\theta \qquad x^2 = 64\sec^2\theta$$

$$= \int \frac{1}{64}\cos\theta\,d\theta$$

$$= \frac{1}{64}\sin\theta + C$$

$$= \boxed{\frac{\sqrt{x^2-64}}{64x} + C}$$

$$\sqrt{x^2-a^2} \Rightarrow x = a\sec\theta$$



From triangle (**SOH**-CAH-TOA):

$$\sin\theta = \frac{\sqrt{x^2-64}}{x}$$

Integration by Partial Fractions

- Used for integrals of the form $\int \frac{P(x)}{Q(x)}\,dx$
  - Where degree of Q(x) > P(x)

Partial fraction decomposition table

Type	*Factor example*	*Decomposition*
Linear factor	$(x-4)$	$\frac{A}{x-4}$
Repeated linear factor	$(x-4)^2$	$\frac{A}{(x-4)}+\frac{B}{(x-4)^2}$
Quadratic irreducible factor	(x^2+4)	$\frac{Ax+B}{(x^2+4)}$
Repeated quadratic irreducible factor	$(x^2+4)^2$	$\frac{Ax+B}{(x^2+4)}+\frac{Cx+D}{(x^2+4)^2}$

Integration by Partial Fractions - Example

- Integrate $\int \frac{2x^2+3x+1}{x^3-x^2+4x-4}\,dx$ $\xrightarrow{\text{Factor denom.}}$ $\int \frac{2x^2+3x+1}{(x-1)(x^2+4)}\,dx$

Linear

Irreducible quadratic

So, $\frac{2x^2+3x+1}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$ $\xrightarrow{\text{Mult. everything by original denom.}}$ $2x^2+3x+1 = A(x^2+4) + (Bx+C)(x-1)$

$= Ax^2 + 4A$

$= Bx^2 - Bx + Cx - C$

$= Bx^2 + (C-B)x - C$

Group like terms (x², x, constants)

$\underline{(A+B)}x^2 + \underline{(C-B)}x + \underline{(4A-C)}$ $\xrightarrow{\text{Compare with numerator}}$ $\underline{2}x^2 + \underline{3}x + \underline{1}$

We get the system:

$$A+B=2$$

$$C-B=3$$

$$4A-C=1$$

$B = 2 - A$

$C - (2-A) = 3 \Rightarrow C = 5 - A$

$4A - (5-A) = 1$

$5A - 5 = 1 \Rightarrow A = \frac{6}{5}$

$B = 2 - \frac{6}{5} = \frac{4}{5}$

$C = 5 - \frac{6}{5} = \frac{19}{5}$

$\int \left(\frac{6/5}{x-1} + \frac{(4/5)x + 19/5}{x^2+4} \right) dx$

Simplify and integrate from here

Integration by Partial Fractions – Example continued

From last slide:

$$\int \left(\frac{6/5}{x-1} + \frac{(4/5)x + 19/5}{x^2+4}\right) dx \quad = \frac{1}{5}\int \left(\frac{6}{x-1} + \frac{4x+19}{x^2+4}\right) dx$$

$$= \frac{1}{5}\left[\int \frac{6}{x-1}dx + \int \frac{4x}{x^2+4}dx + \int \frac{19}{x^2+4}dx\right]$$

arctan case:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$= 6\ln|x-1|$$

u-sub with

$u = x^2+4$, $du = 2x\,dx$

$$= 2\ln(x^2+4)$$

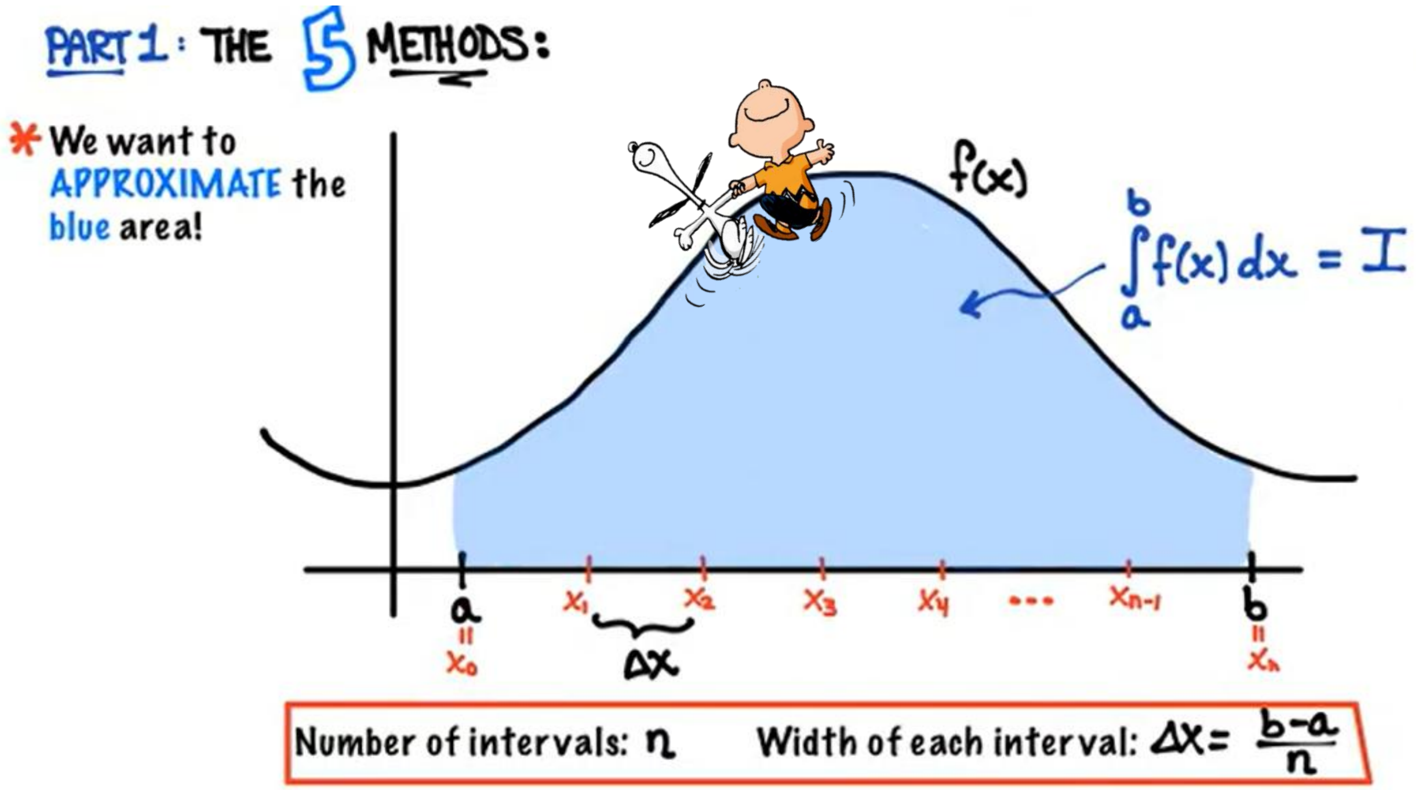
$$= \frac{19}{2}\tan^{-1}\left(\frac{x}{2}\right)$$

$$\boxed{\frac{1}{5}\left[6\ln|x-1| + 2\ln(x^2+4) + \frac{19}{2}\tan^{-1}\left(\frac{x}{2}\right)\right] + C}$$

Note: This example is as bad as it gets, you shouldn't see anything this hard on an exam!

Numerical Integration

- Used to approximate integrals (aka area under the curve)
  - 5 methods → LHS, RHS, Midpoint, Trapezoidal, Simpson's



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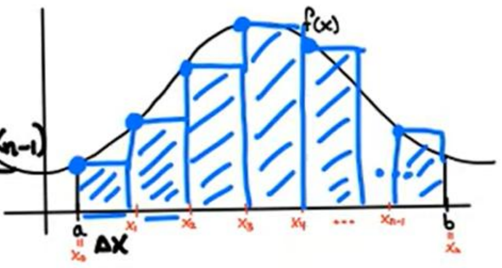
Numerical Integration – LHS & RHS

METHOD #1: [LEFT-HAND SUM] "L_n" ← # of RECTANGLES.

$$L_n = \underbrace{\Delta x \cdot f(x_0)}_{\text{REC \#1}} + \underbrace{\Delta x \cdot f(x_1)}_{\text{REC \#2}} + \dots + \underbrace{\Delta x \cdot f(x_{n-1})}_{\text{REC \#n}}$$



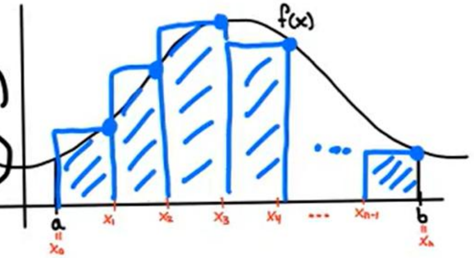
$$L_n = \Delta x \left(\underbrace{f(x_0) + f(x_1) + \dots + f(x_{n-1})}\right)$$

- f(x) increasing → Underestimate
- f(x) decreasing → Overestimate

METHOD #2: [RIGHT-HAND SUM] "R_n"

$$R_n = \Delta x \cdot f(x_1) + \Delta x \cdot f(x_2) + \dots + \Delta x \cdot f(x_n)$$

$$R_n = \Delta x \left(\underbrace{f(x_1) + f(x_2) + \dots + f(x_n)}_{n}\right)$$



- f(x) increasing → Overestimate
- f(x) decreasing → Underestimate

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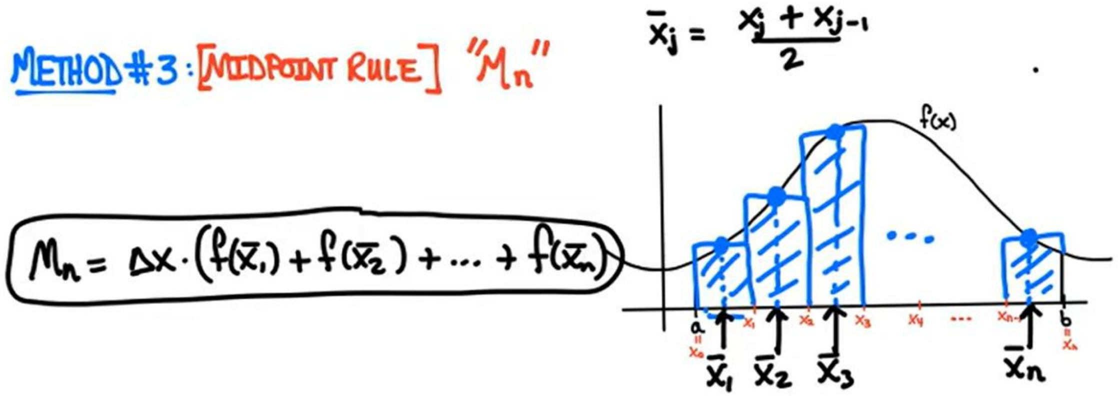
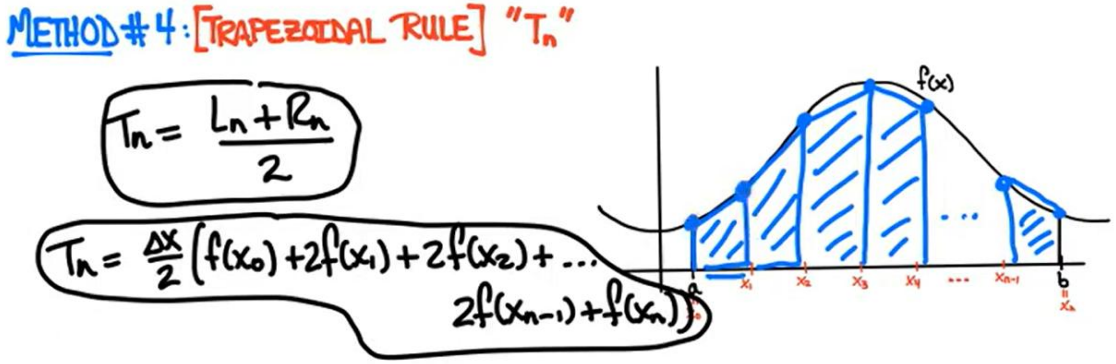
Numerical Integration – MR & TR



- f(x) concave up → Underestimate
- f(x) concave down → Overestimate



- The average of LHS and RHS
- f(x) concave up → Overestimate
- f(x) concave down → Underestimate

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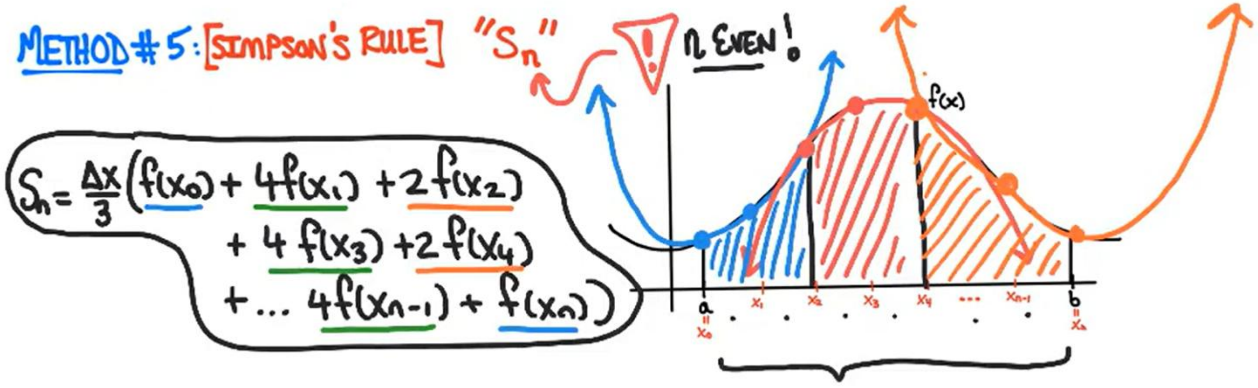
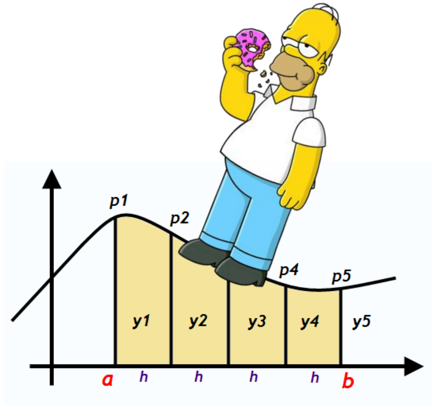
Numerical Integration – Simpson's Rule





$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(x_0) + 4\sum_{i=1,3,5,\dots}^{n-1} f(x_i) + 2\sum_{i=2,4,6,\dots}^{n-2} f(x_i) + f(x_n)\right]$$

Pattern to memorize:

- First + last → coefficient **1**
- Odd indices → coefficient **4**
- Even indices → coefficient **2**

- Area under multiple parabolas
- Typically most accurate method
- Uses 3 points per parabola (left, mid, right)

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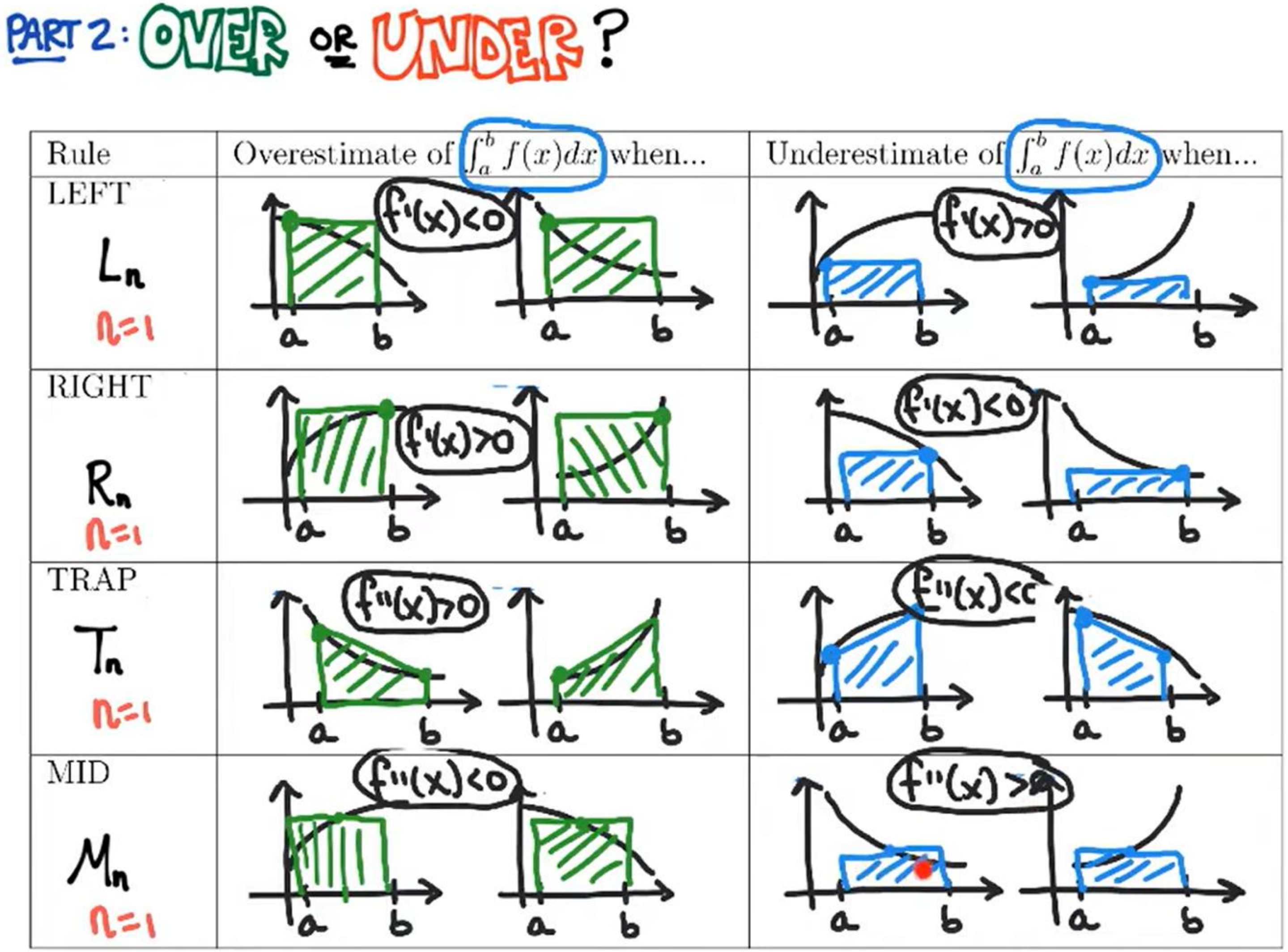
Numerical Integration – Over or Underestimate?



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Numerical Integration – Error

PART 3: ERROR IN APPROXIMATIONS of $\int_a^b f(x)\,dx$

$$\boxed{E_T = \underbrace{\int_a^b f(x)\,dx}_{\text{EXACT}} - \underbrace{T_n}_{\text{APPROX}}} \qquad \boxed{E_M = \int_a^b f(x)\,dx - M_n} \qquad \boxed{E_S = \int_a^b f(x)\,dx - S_n}$$

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \qquad |E_M| \leq \frac{K(b-a)^3}{24n^2} \qquad |E_S| \leq \frac{K(b-a)^5}{180n^4}$$

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Numerical Integration – Simpson's with Error example

- Approximate $\int_0^1 e^x \, dx$ using Simpson's Rule with n = 4 and find an error bound

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

Where K = the maximum value of the fourth derivative of f(x) on [0,1]

$$\left.\begin{array}{l} b = 1 \\ a = 0 \\ K = e \\ n = 4 \end{array}\right\} \text{Plug into formula} \longrightarrow \boxed{|E_S| \leq \frac{e(1)^5}{180(4^4)}}$$

*fine to leave as is, after simplification you get 0.000059

Great, what does this mean?

This means that our approximation using Simpson's Rule is within 0.000059 of the true value, with n = 4

n = 4:

$$f'(x) = e^x$$

$$f''(x) = e^x$$

$$f^{(3)}(x) = e^x$$

$$f^{(4)}(x) = e^x \longrightarrow$$

f''''(0) = e^0 = 1
f''''(1) = e^1 = e → e is the max on [0,1], so **K = e**

Improper Integration

- Improper integral: An integral that either has an **infinite interval** or an integral that **“blows up” infinitely** somewhere due to a **discontinuity**
  - Key idea → Break up integral into 2 parts and rewrite as limits!
  - 2 types:

Type I: Infinite interval

$$\int_a^\infty f(x)\,dx \quad \text{or} \quad \int_{-\infty}^\infty f(x)\,dx$$

Infinite upper bound:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

Infinite lower bound:

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

Type II: Discontinuity (vertical asymptote)

$$\int_a^b f(x)\,dx \quad \text{where } f(x) \to \infty \text{ at some point in } [a, b]$$

Discontinuity at endpoint:

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

Discontinuity inside interval:

Split it:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

and take limits on both sides.

Improper Integration – Infinite interval example

- Evaluate $\int_1^{\infty} \frac{1}{x^2}\,dx$

The infinity in the upper bound is an issue, so we have to **rewrite it as t**, and take the limit of the integral as **t approaches infinity**

$$\int \frac{1}{x^2}dx = \int x^{-2}dx = -\frac{1}{x}$$

Rewriting the integral with a limit and b = t, we get:

$$\lim_{t\to\infty} \overbrace{\int_1^t \frac{1}{x^2}\,dx}$$

So, $\lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left(\underbrace{-\frac{1}{t}}+1\right)$ = 1

As t → infinity, the left term in the parentheses goes to zero

Therefore,

$$\boxed{\int_1^{\infty} \frac{1}{x^2}\,dx = 1}$$

Improper Integration – Discontinuity example

- Evaluate $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

The square root function in the denominator is an issue at x = 0, so we must **rewrite 0 as t** and take the limit of the integral as **t approaches zero**

$$\int x^{-1/2}dx = 2\sqrt{x}$$

Rewritten: $\lim_{t\to 0^+} \overbrace{\int_t^1 \frac{1}{\sqrt{x}}\,dx}$

(from the **positive side only** to avoid imaginary numbers)

So, $\lim_{t\to 0^+} [2\sqrt{x}]_t^1 = \lim_{t\to 0^+}(2 - \underbrace{2\sqrt{t}}_{= 0})$ = 2

Therefore,

$$\boxed{\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2}$$

Good luck studying!