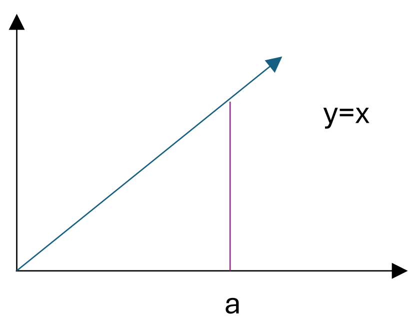
**Chapter 15 Review – Multiple Integrals**

Hello everyone, it's Akeva! This resource is going to be going over how to go about solving problems with multiple integrals, change of coordinates (and when to use them), and the meanings of integrals. Let's jump right into each section and the key takeaways you should to for the exam, whether it be definitions or applications to solving problems. Once again, this resource can help you put together your cheat sheet and, hopefully, explain more some concepts you are unsure of.

**Double Integrals over Rectangles and General Regions**

What do double integrals mean?

In this section, y'all were introduced to the concept of a double integral. Let's go back to a Calculus 1 concept that hopefully will make sense to you if you're unsure what a double integral does. Remember how we learned that a single integral allowed us to find the area under a given curve. For example, $\int xdx = \frac{1}{2}x^2 + C$. If we think about it, it's the equation for finding the area of a triangle with equal base and height, which y=x alludes to. For a visual, let's look at a graph.




Looking from 0 to a on the x-axis, we can see that x = a, y = a, revealing an isosceles right triangle (base and height equal). To find the area under y = x from 0 to a, that's the integral defined above. With 0 and a serving as our bounds, we get $\frac{1}{2}(2^2 - 0^2) = \frac{1}{2}(4) = \frac{4}{2} = 2$. If we take the standard equation for finding the area of a triangle, we get $\frac{bh}{2}$ where b=h=2. This leads to the same result.

Awesome, so how can we use this concept in double integrals? Well, if a single integral finds the area of a univariable function, then a double integral of a bivariate function finds the volume under a surface defined by x and y, for example. Taking it a step further and foreshadowing our discussion of triple integrals, this means taking a triple integral can help determine the volume within 3D regions through a trivariate function. Essentially, we go into a higher dimension that what is defined by the function. For a single integral, 1D to 2D. For a double integral, 2D to 3D and the same for a triple integral (in the way I think about it). Now why is that? Because f (x, y) and f(x,y,z) are essentially describing the surface. In the case for bivariate functions, the surface hovers over another axis. In trivariate functions, the surface encompasses all three axes, so we are interested in what's inside as opposed to what's underneath. If I knew how to draw in 4D, it would be like the above example where we become interested in what is under the shape and on the 4$^{th}$ axis.

<u>Definitions of Double Integrals</u>

Now that we can contextualized double integrals, let's go over some equations.

$$\iint f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(xij^{*}, yij^{*})\Delta A$$

$$V = \iint_{R} f(x,y)dA$$

The first equation communicates the same idea as the second but using a double Riemann sum and serves an approximation. We did something similar in Calculus 1 when we first defined integrals as rectangles under a curve. Both are defined for some region, defined as R. For an example on how to use the first formular to solve an example, refer to this video by Krista King. With regards to the second equation, that is the form that I am sure y'all have been using for the most part. For a quick recap, let's break down what's going on in the second equation:

1. dA = refers to the defined region we are taking the double integral of. Remember that this only works for bivariate functions. So, however, the function is defined, its variables will show up here. With the above equation dA = dx,dy, without any specific order.
2. The integrals themselves = the integrals should appear in the inverse order as they do in dA. This means if dx comes before dy, the second integral from left to right should be the bounds for the x axis. This leads to an iterated integral. For a visual, look below:

$$V = \int_{c}^{d} [\int_{a}^{b} f(x,y)dx]\, dy$$

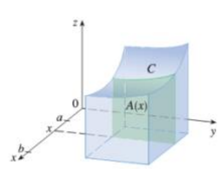



FIGURE 11

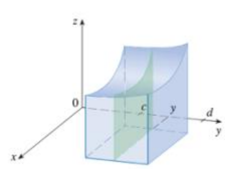


FIGURE 12

For those who are curious, the figures come from your textbook.

Now the next natural question maybe, well, can I switch the bounds? Because sometimes it can be a pain to integrate x first but is easier for y. The answer is yes; you can switch the bounds ONLY IF f is continuous over the region you are looking to integrate over. If that is the case, you would switch the integrals and the order of dA (remember, if you don't, you will technically be integrating over a different region entirely; it's like how you have to square both sides of an equation or divide both sides by the same factor or else you are solving a different problem.

<u>What are some other ways to calculate the volume and some hacks?</u>

To take things further, if instead of being given the function for the surface but rather a function for the area, you can use this to find the volume directly and is the same as the double integral. It would look something like this:

$$V = \int_a^b A(x)dx$$

For some more examples outside of what was given in class, refer to these videos:

- Organic Chemistry Tutor
- Krista King (features Type I and Type II Regions)
    - Second video of hers
- Differentiated Calculus (Type I and Type II Regions)

For a quick refresher on Type I and II regions, look to this website.

Furthermore, as a trick to make your life easier when it applies, you can separate the x and y axes and multiply their results. This ONLY works if you can define a function completely in x or completely in y. Let's test if you know when you can do it:

1. $xy^2$
2. $x + y$
3. $x + xy$

Which number(s) can be separated?

The answer is that you can only do it for 1 and 3. For 1, there is plainly multiplication between x and y squared. However, for 3, we need to a little bit of manipulation. There is a common factor between the two terms: x. Therefore, it can be written as the following: $x + xy = x(1 + y)$. This would make the function defined x be only x and y would be y + 1. If the bivariate function is separable, you can use the following form:

$$\iint_R g(x)h(y)dA = \int_a^b g(x)dx \int_c^d h(y)dy \ where \ R = [a, b]x[c, d]$$

There are some more properties of double integrates we can make use of:

- $\iint_D [f(x,y) + g(x,y)]\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$
  - Now you may wonder when we use this equation. Think of it as a resource to use when you have terms that technically can't be integrated the same or just a way to take things term by term. Notice that you still term the double integral on each chunk you do.
- Just as with single integrals, any constant can be brought outside the double integrals and be multiplied to the final product of the double sum at the end.
- If $f(x,y) \geq g(x,y)\ for\ all\ (x,y)\ in\ D, then\ \iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA$
  - This can help you gauge your work in terms of knowing that the result of f(x,y) should be larger than g(x,y)
  - This could also apply for when g(x, y) is just a constant
- If D = $D_1 \cup D_2$ where the subsets only share a border, then $\iint_D f(x,y)\, dA =$ $\iint_{D1} f(x,y)\, dA + \iint_{D2} f(x,y)\, dA$
  - This is useful for when the regions in D are neither Type I nor Type II but can be summarized as a sum

Average Value

Previously, we have looked at finding the average value of a function over a particular interval. We can also do the same in Multi. It is defined as:

$$favg = \frac{1}{A(R)} \iint_R f(x,y)\, dA$$

An example of this kind of problem can be found here by Krista King.

**Double Integrals in Polar Coordinates**

Fundamental Equations for Polar Coordinates

- $r^2 = x^2 + y^2$
  - This comes from the fact that the basis or polar coordinates for shapes consisting of circles or subsets of circles such as an arc or wedge
- $x = r\cos\theta$
  - This comes from the fact that cosine corresponds to the x-value of a point in the Unit Circle
    - If you need a refresher, look back to the Chapter 14 Review
- $y = r\sin\theta$

- This comes from the fact that sine corresponds to the y-value of a point in the Unit Circle
- dA coming for Cartesian coordinates into Polar Coordinates is $r\ dr\ d\theta$
    - Do not forget the r because it is imperative to changing coordinate systems. I've done problems where sometimes I forgot it and go the completely wrong answer and/or could not integrate without it.

I really do want to stress that if you are shaky on the Unite Circle, now is the time to review because from now onward, we will be using a lot from it. You only need to memorize the first quadrant and then add negatives to the corresponding quadrants.

Also, many conclusion/properties mentioned in the previous section also apply to polar coordinates. To not make the document too long, I'll omit this.

Problems

When it comes to these problems, you most likely will not be told to use polar coordinates (unless they are specifically testing for this) or be given the problem in polar coordinates already. However, some key signs for you to use polar coordinates is when is when you see a dome, a sphere (yes, you can use polar coordinates for spheres), arcs, clovers, etc. Immediately convert everything into polar coordinates using the above equations and change of variables. Simply the equation. For setting the bounds, and when to use a number or term with another variable, draw lines ask yourself if the boundaries are defined simply by a number such as 0 or 1 or any it only be described by a function. The answers should serve as your bounds. For extra problems on polar coordinates, refer to the following resources:

- Organic Chemistry Tutor
- Professor Dave Explains
- Krista King

**Applications of Double Integrals**

If you are interested on the applications of Double Integrals, refer to this video by Alexandria Niedden. I think she does a good job of breaking down concepts and these concepts will especially come back for my engineering students out there in Statics, Physics, Probability, and Electrical Engineering.

**Surface Area**

There are three essential equations that come out of this topic. They are the following:

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta Tij \ \ where\ \Delta Tij\ tangent\ plane\ to\ approximate\ Sij$$

$$A(S) = \iint_D \sqrt{[fx(x,y)]^2 + [fy(x,y)]^2 + 1}\, dA$$

$$A(S) = \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

The first equation is a representation of the surface area using a double Reimann sum. $\Delta Tij$ is part of a tangent plane that lies above the $Rij$ which is the shadow of the surface ($\Delta Sij$) on the xy plane. The second equation is a surface z = f(x,y) where fx and fy are continuous. The third equation is simply the rewritten form of the second formula.

For practice problems, use the following resources:

- Krista King
- Alexandria Niedden

## Triple Integrals

Fundamental Equations - Cartesian

$$\iiint_B f(x,y,z)dV = \lim_{i,m,n\to\infty} \sum_{i=1}^{i} \sum_{j=1}^{j} \sum_{k=1}^{k} f(xijk^*, yijk^*, zijk^*)\Delta V$$

Where $B = \{(x,y,z) |\ a \le x \le b, c \le y \le d, r \le z \le s\}$

Similarly to previous sections, you are less likely to use this form but is useful to remember that integrals are essentially Reimann sums. The integral form can be seen below.

$$\iiint_B f(x,y,z)\, dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\, dx\, dy\, dz$$

The equation for a triple integral over a general bounded region E can be shown below:

$$\iiint_E f(x,y,z)\, dV = \iiint_B F(x,y,z)\, dV$$

$$\iiint_E f(x,y,z)\, dV = \iint_D \left[\int_{u1(x,y)}^{u2(x,y)} f(x,y,z)dz\right] dA$$

Where $E = \{(x, y, z) | \ (x, y) \ \in D, \ u1(x, y) \leq z \leq u2(x, y)\}$

For Type I regions:

$$\iiint_E f(x, y, z) dV = \int_a^b \int_{g1(x)}^{g2(x)} \int_{u1(x,y)}^{u2(x,y)} f(x, y, z) \, dz \, dy \, dx$$

For Type II regions:

$$\iiint_E f(x, y, z) \, dV = \int_c^d \int_{h1(x)}^{h2(x)} \int_{u1(x,y)}^{u2(x,y)} f(x, y, z) \, dz \, dx \, dy$$

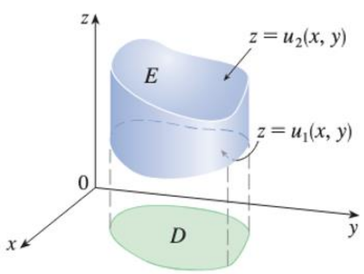



**FIGURE 2**
A type 1 solid region

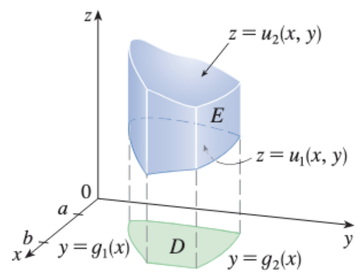



**FIGURE 3**
A type 1 solid region where the projection *D* is a type I plane region

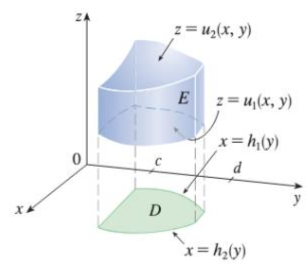



**FIGURE 4**
A type 1 solid region with a type II projection

(from the textbook)

When the projection is on the yz axis, integrate with respects to x first using u1(y,z) and u2(y,z) as the bounds. Conversely, if the projection is on the xz axis, differentiate with respect to the missing variable, y first.

For additional problems of triple integrals in Cartesian only, use the following videos:

- Krista King
- Alexandria Niedden
- Organic Chemistry Tutor
- Michel van Biezen
  - For a full playlist of triple integrals with all the coordinate systems use this link

Fundamental Equations – Cylindrical

To go from cylindrical to Cartesian coordinates use the following equations:

$$x = rcos\theta$$

$$y = rsin\theta$$

$$z = z$$

Note that it's exactly like polar coordinates by with a z-axis. This because polar is best for circles and when you extrude a circle up or down, you get a cylinder.

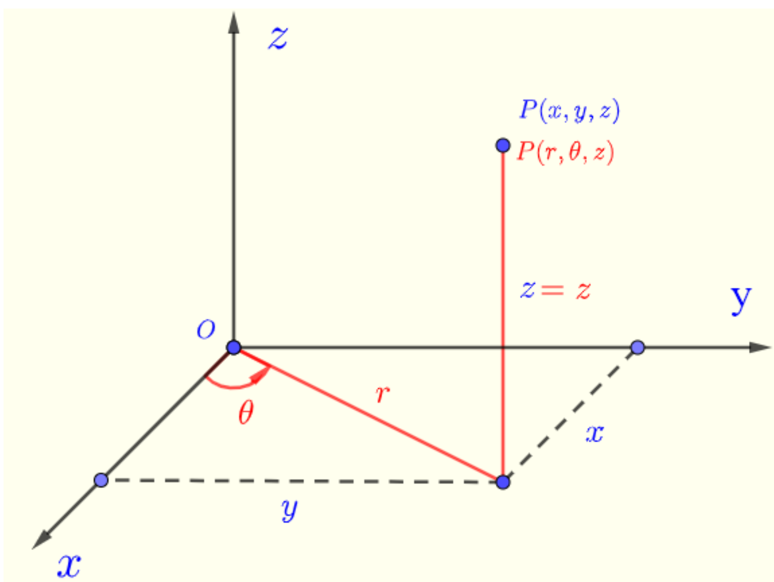
To go from Cartesian to cylindrical, use the following equations:

$$r^2 = x^2 + y^2$$

$$tan\theta = \frac{y}{x}$$

$$z = z$$

You may be needing a visual of this conversion. Look at the figure below.




Source

To write it into words, theta is the angle from the x-axis to the y-axis, r is the radius to the point on the xy plane and z does not change. As one infers, this coordinate system is most helpful for cylindrical shapes (do you remember what those equations look like? If not, go back and review).

Similarly to the polar coordinates, you need an extra conversion factor for cylindrical coordinates:

dA = $r\ dz\ dr\ d\theta$

For more practice problems with cylindrical coordinates, refer to the following videos:

- Krista King
- Alexandra Niedden
- Refer to the playlist I listed above for more examples with cylindrical coordinates

Fundamental Equations – Spherical

I will spend a little bit more time defining the variables here because I remember being really confused when I first learned spherical coordinates. The three variables used in the coordinate system are $\rho$ $(roe), \theta, and\ \varphi$ $(phi)$. $\rho$ acts as r does in polar and cylindrical coordinates in that it's the radius. However, as the name suggests for spherical coordinates, you can draw the radius anywhere on the surface of a sphere and different angles which is why we have the other two variables. $\theta$ is defined the same way as in cylindrical coordinates in that is the angle of the point between the x-axis and y-axis (I should say that 0 is defined as being right on the x-axis). $\varphi$ is really what differentiating spherical from cylindrical coordinates. It's an extra angle off the z-axis that is bounded between 0 and $\pi$. You think of it picking up the point off the xy plane and having it float in the air. With this explanation, see if the figure below makes more sense if you were confused.

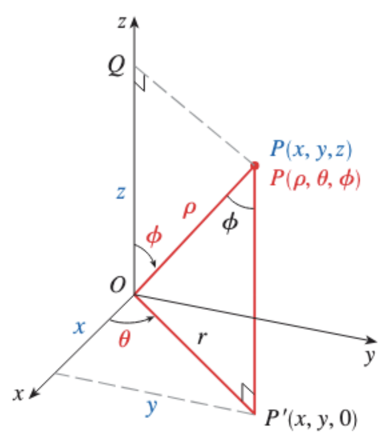



**FIGURE 5**

If not, try referring to this video.

To go from spherical coordinates to the Cartesian coordinate, use the following equations (note the commonality in x and y of the sin of phi):

$$x = \rho\ sin\varphi\ cos\theta$$

$$y = \rho\ sin\varphi\ sin\theta$$

$$z = \ \rho\ cos\ \varphi$$

$$\rho^2 = x^2 + y^2 + z^2$$

Also note that $\rho^2$ is defined similarly to how $r^2$ is defined but with the extra axis of z since we are working with 3D shapes.

To go from spherical coordinates to cylindrical coordinates, use the following equations:

$$z = \rho \cos \varphi$$

$$r = \rho \sin \varphi$$

Now, a conversion factor is also needed. In the case of the spherical coordinate, dA is defined as

$$\rho^2 sin\varphi$$

Some additional problems can be found below:

- Krista King
- Alexandria Niedden
- Playlist listed above in the beginning section of Triple Integrals

Change of Variables in Multiple Integrals

Those extra conversion factors we had to put when changing coordinates have a formal called Jacobians. This are particularly helping when doing a transformation allows for easier calculations.

Image: the output of a transformation T whose subsets are in all real numbers.

To get the Jacobian, which gives us the image of a point or line of a bivariate function, we find the determinant which is defined as:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

(note: if you have to take Linear Algebra, these concepts will come back)

Like how we do for the polar, cylindrical, and spherical coordinate system conversions, you just put the calculated Jacobian on at the end with du dv. If no two points have the same image, the transformation is called one-to-one (another concept that comes up in Linear Algebra).

For a trivariate function, you do something very similar. To make things cleaner, I will use filler letters to communicate how to calculate the determinant and then the determinant with the partial derivatives.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

To get used to these, I would suggest doing a lot of practice problems. Use the following videos as ways to practice:

- Alexandria Niedden
- Professor Leonard
- Dr. Trefor Bazett

Alright! This concludes a general review of Chapter 15! Good luck studying!